The lattice of all clones of self-dual functions in three-valued logic

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The lattice of all clones of self-dual functions in three-valued logic is described. Even though this lattice contains a continuum of clones, a simple description was found. Using this description various properties of the lattice and of the clones were derived. Pairwise inclusion of the clones into each other was described, and bases for all clones were found. Also, for each clone the relation degree, the cardinalities of the corresponding principal filter and principal ideal were determined.

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Preamble

This paper is devoted to the classical problem of Clone Theory: finding a description of the lattice of clones. In [10], [11] Post described all clones in two-valued logic. It turned out that all such clones are finitely generated and the lattice of these clones is countable. But in 1959 it was proved that there exists a continuum of clones in k-valued logic for $k \geq 3$ [6]. Hence, it seems hardly possible to obtain a complete description of the lattice of all clones even in three-valued case. Nevertheless, Jablonskij [4] described all maximal (also known as precomplete) clones in three-valued logic. It turned out that all maximal clones except the clone of all linear functions contain a continuum of subclones [3, 8].

This paper is devoted to the maximal clone of self-dual functions. It consists of all functions that preserve the relation $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$. Important results on this subject were obtained by Marchenkov. He and co-authors found many clones of self-dual functions [9] and showed that there exists a continuum of such clones [8].

In spite of continuum cardinality we found a complete description of all clones of self-dual functions, which is presented in this paper. Thus, this is the first maximal clone besides the clone of all linear functions that has such description.

In the paper we define a set of predicates Π (we do not distinguish sharply between relations and predicates; since we operate with formulas, it is usually proper for us to use the word predicate). Using these predicates we define a class of clones Υ that has continuum cardinality. These are all clones of self-dual functions except countably many clones. The other clones are divided into two classes Θ and Φ. The finite class Θ consists of all clones that are self-dual with respect to the permutation of 0 and 1. The countable class Φ contains all remaining clones. Note that all clones in the finite class Θ and many clones in the countable class Φ were already found in [9]. In the definition of the class Φ we say precisely which clones are already known from [9] and which are new. Thus, the main result of this paper is the description of the class Υ.

Every clone is defined as the set of all functions that preserve some set of relations (finite or infinite). We find the relation degree for every clone, and thereby prove that our description is minimal.

Using the description we show various properties of the lattice and of the clones. It is well-known that some clones in three-valued logic have no basis [6]. Nevertheless, we prove that every clone of self-dual functions has a basis (finite or infinite), and we present bases for all of them.

We also describe pairwise inclusion of the clones into each other. For the finite and countable classes of clones pairwise inclusion is shown by a graph in Figure 2. For the class of clones Υ we formulate theorems that describe pairwise inclusion. Finally, for each clone in the lattice we find the cardinalities of the corresponding principal filter and principal ideal.

As expected, the description of the class Υ is rather complicated. Nevertheless, all listed properties can be easily derived from it. Moreover, in the first section we show that using our description we can easily obtain every finite sublattice of the part of the lattice that has continuum cardinality. We present the lattice of all clones that can be defined by predicates of arity 4. This lattice was found without a computer. Of course, using a computer bigger sublattices can be completely described.

To obtain the main result we essentially use the Galois connection between clones and relational clones. Moreover, we do not use the closure operator for functions, and functions are only auxiliary objects in the paper. It can be said that we find relational clones that contain the relation $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$.

The main idea of the proof is the following. We introduce a notion of an essential predicate. Essential predicates are all predicates that can not be presented as a conjunction of predicates with smaller arities. A closure operator is defined on the set of all essential predicates, and it is proved that there exists a one-to-one correspondence between relational clones and closed sets of essential predicates. Thus, to describe all clones we just need to describe all closed sets of essential predicates. The set of all essential predicates is small enough, and for most clones of self-dual functions we completely describe all essential predicates that are preserved by functions from this clone.

This paper is organized as follows. In Section 1 we give main definitions and formulate main results of the paper. There we define three classes of clones, describe pairwise inclusion of clones into each other, and present bases for all clones. At the end of this section we formulate theorems that for every clone determine the relation degree, the cardinalities of the corresponding principal filter and principal ideal.

In Section 2 we introduce necessary notions and prove important properties related to these notions. There we define a closure operator for predicates and the Galois connection between clones and relational clones. Then, necessary notations are described. After that we formulate the notion of an essential predicate and prove various properties of essential predicates. At the last part of this section we define the closure operator on the set of all essential predicates and prove important properties of this closure operator.

Section 3 is devoted to the construction of the classes Φ and Υ. Firstly, we describe all essential predicates that are preserved by the self-dual extension of disjunction. Then, we sequentially construct the lattice of the clones. Finally, we prove that if a clone contains the extension of disjunction and preserves the set $\{0, 1\}$, then this clone belongs to Φ, Υ, or Θ.

In Section 4 we prove the main statements and theorems of this paper. Firstly, we show that clones from $\Theta \cup \Phi \cup \Upsilon$ are all clones of self-dual functions in three-valued logic. Then we prove theorems about pairwise inclusion of clones into each other, theorems related to bases of clones, the relation degree of clones, and other statements.

At the end of the paper we give a list of main notations.

Note that the preliminary version of this result was already published in the book [12] in Russian.

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1 MAIN STATEMENTS AND THEOREMS.

1.1 Main definitions.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $E_k = \{0, 1, 2, \dots, k-1\}$, for $n \in \mathbb{N}$

$$P_k^n = \{f \mid f : E_k^n \to E_k\}, \quad P_k = \bigcup_{n \geq 1} P_k^n.$$

Suppose $F \subseteq P_k$, then by $[F]$ we denote the closure of F under superposition [7]. A set $F \subseteq P_k$ is called a *clone* if F is closed and F contains all projections. By J_k we denote the set of all projections. The clones form an algebraic lattice whose least element is J_k and whose greatest element is P_k.

A mapping $E_k^h \to \{0, 1\}$ is called an *h-ary predicate*. For $h \in \mathbb{N}_0$ let

$$R_k^h = \{\rho \mid \rho : E_k^h \to \{0, 1\}\}, \quad R_k = \bigcup_{h \geq 0} R_k^h.$$

As mentioned above, we do not distinguish between predicates and relations. So instead of $\rho(a_1, \dots, a_n) = 1$ we also write $(a_1, \dots, a_n) \in \rho$. Sometimes we write $a_1 a_2 \dots a_h$ instead of $(a_1, a_2, \dots, a_h)$ and operate with tuples like with words. Let $\alpha \in E_k^h$, then by $\alpha(i)$ we denote the i-th element of α. We suppose that functions from P_k are also defined in the usual way on tuples or words from E_k^h. That is, suppose $\alpha_1, \dots, \alpha_n \in E_k^h$, $f \in P_k^n$, then we put $f(\alpha_1, \dots, \alpha_n) = \beta$, where $\beta \in E_k^h$, $\beta(i) = f(\alpha_1(i), \alpha_2(i), \dots, \alpha_n(i))$ for every $i \in \{1, 2, \dots, h\}$.

In this paper predicates are often written as matrices. We write

$$\rho = \begin{pmatrix} b_{1,1} & b_{2,1} & \dots & b_{n,1} \\ b_{1,2} & b_{2,2} & \dots & b_{n,2} \\ \dots & \dots & \dots & \dots \\ b_{1,h} & b_{2,h} & \dots & b_{n,h} \end{pmatrix}$$

if $\rho \in R_k^h$, $\rho(b_{i,1}, b_{i,2}, \dots, b_{i,h}) = 1$ for every $i \in \{1, 2, \dots, n\}$ and the predicate ρ is equal to 0 on the other tuples.

We say that a function $f \in P_k^m$ *preserves a predicate* ρ if

$$f(\alpha_1, \alpha_2, \dots, \alpha_m) \in \rho$$

for every $\alpha_1, \alpha_2, \dots, \alpha_m \in \rho$.

By $\mathrm{Pol}(\rho)$ we denote the set of all functions $f \in P_k$ that preserve a predicate ρ. For $S \subseteq R_k$ we put $\mathrm{Pol}(S) = \bigcap_{\rho \in S} \mathrm{Pol}(\rho)$.

By $\mathrm{Inv}(f)$ we denote the set of all predicates $\rho \in R_k$ that are preserved by a function f. For $M \subseteq P_k$ we put $\mathrm{Inv}(M) = \bigcap_{f \in M} \mathrm{Inv}(f)$.

Let $\sigma : E_3 \longrightarrow E_3$, $\sigma(0) = 1$, $\sigma(1) = 0$, $\sigma(2) = 2$. For $\rho \in R_3$ by ρ^* we denote the predicate that is dual to ρ with respect to the transposition σ:

$$\rho^*(x_1, \dots, x_n) := \rho(\sigma(x_1), \sigma(x_2), \dots, \sigma(x_n)).$$

Note that this duality is not the same as the duality in the definition of self-dual functions, which is the duality with respect to the cyclic permutation of 0,1 and 2. Suppose $S \subseteq R_3$, then put $S^* := \{\rho^* \mid \rho \in S\}$.

1.2 The lattice of the clones

Now we define several predicates, which we are going to use to define the three classes of clones.

$$\rho_{+1} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix},\ \rho_{\leq} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$\rho_N = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix},\ \rho_W = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix},$$

$$\rho_T = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix},\ \rho_{\neq,01} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \rho_{\neq} = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 2 & 0 & 1 \end{pmatrix},$$

$$\rho_{x+y+z}(x_1, x_2, x_3) = 1 \Leftrightarrow x_1 + x_2 + x_3 = 0(mod\ 3),$$

$$\rho_{x \oplus y \oplus z}(x_1, x_2, x_3) = 1 \Leftrightarrow \\ (\forall i\ x_i \in \{0, 1\}) \wedge (x_1 + x_2 + x_3 = 0(mod\ 2)),$$

$$\rho_{\vee,n}(x_1, \dots, x_n) = 1 \Leftrightarrow \\ (\forall i\ x_i \in \{0, 1\}) \wedge ((x_1 = 1) \vee (x_2 = 1) \vee \dots \vee (x_n = 1)),$$

$$\rho_{\to,n}(x_1, \dots, x_n, x_{n+1}) = 1 \Leftrightarrow \\ (\forall i\ x_i \in \{0, 1\}) \wedge ((x_1 = 1) \vee \dots \vee (x_n = 1) \vee (x_{n+1} = 0)).$$

In other words, $\rho_{\vee,n} = \{0, 1\}^n \setminus \{0^n\}$, $\rho_{\to,n} = \{0, 1\}^{n+1} \setminus \{0^n 1\}$.

$$\rho_{=,01}(x_1, x_2, x_3) = 1 \Leftrightarrow \\ (x_1 = 1) \vee ((x_1 = 0) \wedge (x_2, x_3 \in \{0, 1\}) \wedge (x_2 = x_3)),$$

$$\rho_{=,012}(x_1,x_2,x_3)=1 \Leftrightarrow (x_1=1)\vee((x_1=0)\wedge(x_2=x_3)).$$

Class Θ of clones.

$$\mathbf{S}=\mathrm{Pol}\left(\rho_{+1}\right),\ \ \mathbf{S_0}=\mathrm{Pol}\left(\{\rho_{+1},\{0\}\}\right),$$

$$\mathbf{SL}=\mathrm{Pol}\left(\{\rho_{+1},\rho_{x+y+z}\}\right),\ \ \mathbf{SL_0}=\mathrm{Pol}\left(\{\rho_{+1},\rho_{x+y+z},\{0\}\}\right),$$

$$\mathbf{1S}=[\{(x+1)(mod\,3)\}]=\mathrm{Pol}\left(\{\rho_{+1},\rho_{\neq}\}\right),\ \ \mathbf{T}=\mathrm{Pol}\left(\{\rho_{+1},\rho_{T}\}\right),$$

$$\mathbf{C}=\mathrm{Pol}\left(\{\rho_{+1},\{0,1\}\}\right),\ \ \mathbf{D}=\mathrm{Pol}\left(\{\rho_{+1},\rho_{\neq,01}\}\right),$$

$$\mathbf{M}=\mathrm{Pol}\left(\{\rho_{+1},\rho_{\leq}\}\right),\ \ \mathbf{DM}=\mathbf{D}\cap\mathbf{M},\ \ \mathbf{DN}=\mathrm{Pol}\left(\{\rho_{+1},\rho_{N},\rho_{N}^{*}\}\right),$$

$$\mathbf{TD}=\mathbf{T}\cap\mathbf{D},\ \ \mathbf{TM}=\mathbf{T}\cap\mathbf{M},\ \ \mathbf{TN}=\mathbf{DN}\cap\mathbf{T},$$

$$\mathbf{L_2}=\mathrm{Pol}\left(\{\rho_{+1},\rho_{x\oplus y\oplus z}\}\right),\ \ \mathbf{TL_2}=\mathbf{L_2}\cap\mathbf{T},$$

$$\mathbf{C_2}=\mathbf{L_2}\cap\mathbf{M},\ \ \mathbf{TC_2}=\mathbf{C_2}\cap\mathbf{T},\ \ \mathbf{J_3}=[\{x\}].$$

Note that all clones from the class Θ were already found in [9].

Relations in Π. We will need the following notation to define the classes Φ and Υ. Let $m\in\mathbb{N}$, $n\in\mathbb{N}_0$. By D_n^m we denote the set of all tuples $(A_1,\ldots,A_m)$ such that $A_1,\ldots,A_m\subseteq\{1,2,\ldots,n\}$, $A_1\cup\ldots\cup A_m=\{1,2,\ldots,n\}$. In case of $n=0$ we have $A_1=A_2=\ldots=A_m=\varnothing$.

Put $D=\bigcup\limits_{m+n\geq 3} D_n^m$. Let us define several binary relations on the set D. Suppose

$$(A_1',\ldots,A_{m'}')\in D_{n'}^{m'},\ (A_1,\ldots,A_m)\in D_n^m.$$

Relation $\simeq$. Let

$$(A_1',\ldots,A_{m'}')\simeq(A_1,\ldots,A_m)$$

iff $m'=m$, $n'=n$, and there exists a permutation $\sigma:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ such that $A_i'=\sigma(A_i)$ for every $i\in\{1,2,\ldots,m\}$.

Relation $\lesssim^1$. Let

$$(A_1',\ldots,A_{m'}')\lesssim^1(A_1,\ldots,A_m)$$

iff $m'\geq m$, $n'\leq n$, $m'+n'=m+n$, $A_i'=A_i\cap\{1,2,\ldots,n'\}$ for $i\in\{1,2,\ldots,m\}$, $A_i'=\varnothing$ for $i\in\{m+1,m+2,\ldots,m'\}$.

Relation $\lesssim^2$. Let

$$(A_1',\ldots,A_{m'}')\lesssim^2(A_1,\ldots,A_m)$$

iff $m' \leq m$, $n' = n$, and the set $\{1, 2, \ldots, m\}$ can be divided into non-overlapping nonempty sets $K_1, K_2, \ldots, K_{m'}$ such that $A'_i = \bigcup\limits_{j \in K_i} A_j$ for every $i \in \{1, 2, \ldots, m'\}$.

Relation $\lesssim^3$. Let

$$(A'_1, \ldots, A'_{m'}) \lesssim^3 (A_1, \ldots, A_m)$$

iff $m' = m$, $n' = n$, $A'_i \supseteq A_i$ for every $i \in \{1, 2, \ldots, m\}$.

Relation $\lesssim$. Suppose $\Omega, \Omega' \in D$, then put $\Omega' \lesssim \Omega$ iff

$$\exists \Omega_1 \exists \Omega_2 \exists \Omega_3 \; (\Omega' \lesssim^3 \Omega_3 \wedge \Omega_3 \lesssim^2 \Omega_2 \wedge \Omega_2 \lesssim^1 \Omega_1 \wedge \Omega_1 \simeq \Omega).$$

The proof of the following lemma is rather simple, but cumbersome. That is why we omit the proof and refer the reader to [12].

Lemma 1.1. *The binary relation $\lesssim$ is transitive and reflexive.*

Hence the binary relation $\lesssim$ determines a quasiorder on the set D.

Note that $(\varnothing, \varnothing, \varnothing) \lesssim (A_1, \ldots, A_m)$ for every $(A_1, \ldots, A_m) \in D$.

To each $(A_1, \ldots, A_m) \in D^m_n$ we assign the predicate $\pi_{A_1, \ldots, A_m} \in R_3^{m+n}$ such that

$$\pi_{A_1, \ldots, A_m}(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1$$

iff the following conditions hold:

1. $\forall i (x_i = 1 \vee (x_i = 0 \wedge (\forall j \in A_i : \; y_j \in \{0, 1\})))$;
2. at least one of the values $x_1, \ldots, x_m, y_1, \ldots, y_n$ is not equal to 0.

It easy to check that $\pi_{\underbrace{\varnothing, \ldots, \varnothing}_{n}} = \rho_{\vee, n}$ for $n \geq 3$.

By Π^m_n we denote the set of all predicates $\pi_{A_1, \ldots, A_m} \in R_3^{m+n}$ such that

$$(A_1, \ldots, A_m) \in D^m_n.$$

Put $\Pi^l = \bigcup\limits_{3 \leq m+n \leq l} \Pi^m_n, \Pi_l = \bigcup\limits_{n \leq l, m+n \geq 3} \Pi^m_n, \Pi = \bigcup\limits_{l} \Pi^l$. It can be easily shown that we have a one-to-one correspondence between elements of D and elements of Π. Then, the binary relations $\simeq, \lesssim^1, \lesssim^2, \lesssim^3, \lesssim$ define the corresponding binary relations on the set Π. For example, the binary relation $\lesssim$ on the set Π is defined as follows

$$\pi_{A'_1, \ldots, A'_{m'}} \lesssim \pi_{A_1, \ldots, A_m} \Longleftrightarrow (A'_1, \ldots, A'_{m'}) \lesssim (A_1, \ldots, A_m).$$

We say that predicates ρ_1 and ρ_2 from Π are *equivalent* if $\rho_1 \lesssim \rho_2$ and $\rho_2 \lesssim \rho_1$. It can be easily checked that two predicates are equivalent iff one can be obtained from another by a permutation of variables. Obviously, the quasiorder $\lesssim$ generates a partial order on the set of the equivalence classes. The quasiorder $\lesssim$ on the set Π up to arity 4 is shown in Figure 1 by a Hasse diagram for the corresponding partial order.

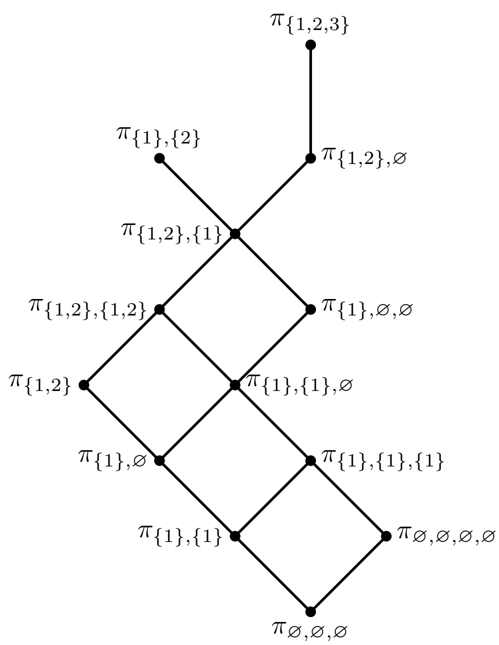

FIGURE 1
Quasiorder on the set Π up to arity 4

We say that a set $F \subseteq \Pi$ is *a downset* if

$$\forall \rho \in F \; \forall \rho' \in \Pi \; (\rho' \lesssim \rho \implies \rho' \in F).$$

By $\widetilde{\Pi}$ we denote the set of all nonempty downsets of Π.

Class Φ of clones.

For $n \geq 2$

$$\mathbf{a_n} = \mathrm{Pol}\left(\{\rho_{+1}, \rho_{\vee,n}\}\right), \quad \mathbf{A_n} = \mathrm{Pol}\left(\left\{\rho_{+1}, \rho^*_{\vee,n}\right\}\right),$$

$$\mathbf{a_n M} = \mathbf{a_n} \cap \mathbf{M}, \quad \mathbf{A_n M} = \mathbf{A_n} \cap \mathbf{M},$$

$$\mathbf{a_n N} = \mathrm{Pol}\left(\{\rho_{+1}, \rho_{\vee,n}, \rho_N\}\right), \quad \mathbf{A_n N} = \mathrm{Pol}\left(\left\{\rho_{+1}, \rho^*_{\vee,n}, \rho^*_N\right\}\right),$$

$$\mathbf{a_\infty} = \bigcap_{i\geq 2} \mathbf{a_i}, \quad \mathbf{A_\infty} = \bigcap_{i\geq 2} \mathbf{A_i},$$

$$\mathbf{a_\infty M} = \bigcap_{i\geq 2} \mathbf{a_i M}, \quad \mathbf{A_\infty M} = \bigcap_{i\geq 2} \mathbf{A_i M},$$

$$\mathbf{a_\infty N} = \bigcap_{i\geq 2} \mathbf{a_i N}, \quad \mathbf{A_\infty N} = \bigcap_{i\geq 2} \mathbf{A_i N},$$

$$\mathbf{aP} = \mathrm{Pol}\left(\{\rho_{+1}, \rho_{\to,2}\}\right), \quad \mathbf{AP} = \mathrm{Pol}\left(\left\{\rho_{+1}, \rho^*_{\to,2}\right\}\right),$$

$$\mathbf{aPN} = \mathrm{Pol}\left(\{\rho_{+1}, \rho_{\to,2}, \rho_N\}\right), \quad \mathbf{APN} = \mathrm{Pol}\left(\left\{\rho_{+1}, \rho^*_{\to,2}, \rho^*_N\right\}\right),$$

$$\mathbf{aP_1} = \mathrm{Pol}\left(\{\rho_{+1}, \rho_{\to,2}, \rho_W\}\right), \quad \mathbf{AP_1} = \mathrm{Pol}\left(\left\{\rho_{+1}, \rho^*_{\to,2}, \rho^*_W\right\}\right).$$

For $n \geq 2$

$$\mathbf{aP_n} = \mathbf{aP_1} \cap \mathrm{Pol}(\pi_{\{1,2,\dots,n\}}), \quad \mathbf{AP_n} = \mathbf{AP_1} \cap \mathrm{Pol}(\pi^*_{\{1,2,\dots,n\}}),$$

$$\mathbf{aP_\infty} = \bigcap_{i\geq 1} \mathbf{aP_i}, \quad \mathbf{AP_\infty} = \bigcap_{i\geq 1} \mathbf{AP_i},$$

$$\mathbf{aQ} = \mathrm{Pol}\left(\{\rho_{+1}, \rho_{=,01}\}\right), \quad \mathbf{AQ} = \mathrm{Pol}\left(\left\{\rho_{+1}, \rho^*_{=,01}\right\}\right),$$

$$\mathbf{aW} = \mathrm{Pol}\left(\{\rho_{+1}, \rho_{=,012}\}\right), \quad \mathbf{AW} = \mathrm{Pol}\left(\left\{\rho_{+1}, \rho^*_{=,012}\right\}\right).$$

Note that many clones from the class Φ were already found in [9]. Precisely, only clones $\mathbf{aP_n}, \mathbf{AP_n}$ for $n \geq 1$, and clones $\mathbf{aP_\infty}, \mathbf{AP_\infty}$, $\mathbf{aQ}, \mathbf{AQ}$ are new.

To define the class Υ, for $F \subseteq \Pi$ we put

$$\mathrm{Clone}(F) = \mathrm{Pol}\left(F \cup \{\rho_{+1}, \rho_W\}\right),$$

$$\mathrm{Clone}^*(F) = \mathrm{Pol}\left(F^* \cup \{\rho_{+1}, \rho^*_W\}\right).$$

Class Υ of clones. Suppose $F \in \widetilde{\Pi}$, then

$$\mathrm{Clone}(F), \mathrm{Clone}^*(F) \in \Upsilon.$$

There are no other clones in Υ.

Theorem 4.4. *Suppose $F_1, F_2 \in \widetilde{\Pi}$, then*

$$\mathrm{Clone}(F_1) \subseteq \mathrm{Clone}(F_2) \Longleftrightarrow F_1 \supseteq F_2.$$

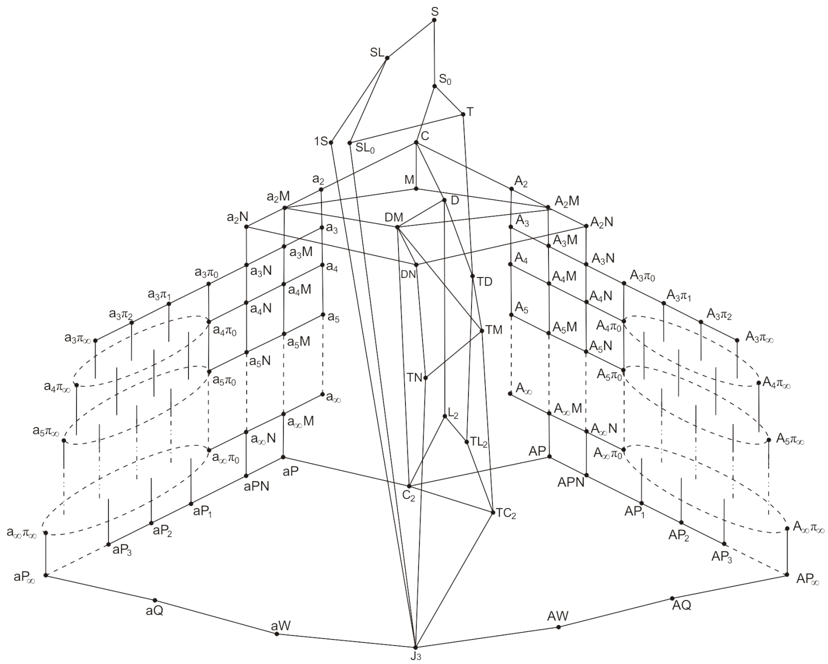

FIGURE 2
The lattice of the clones.

Corollary 1.2. *Suppose $F_1, F_2 \in \widetilde{\Pi}$ and $F_1 \neq F_2$, then*

$$\mathrm{Clone}(F_1) \neq \mathrm{Clone}(F_2).$$

Theorem 4.7. $\Theta \cup \Phi \cup \Upsilon$ *is the set of all clones M such that $M \subseteq \mathrm{Pol}\,(\rho_{+1})$.*

It is hardly possible to draw a lattice that has continuum cardinality on a picture, but we tried to do this in Figure 2. There we draw all clones from the classes Θ and Φ, and also the following clones from the class Υ. For $n \geq 3$ put

$$\mathbf{a_n}\pi_\mathbf{0} = \mathrm{Clone}(\Pi^n \cap \Pi_0), \quad \mathbf{A_n}\pi_\mathbf{0} = \mathrm{Clone}^*(\Pi^n \cap \Pi_0),$$

$$\mathbf{a_n}\pi_\infty = \mathrm{Clone}(\Pi^n), \quad \mathbf{A_n}\pi_\infty = \mathrm{Clone}^*(\Pi^n),$$

$$\mathbf{a}_\infty\pi_\mathbf{0} = \mathrm{Clone}(\Pi_0), \quad \mathbf{A}_\infty\pi_\mathbf{0} = \mathrm{Clone}^*(\Pi_0),$$

$$\mathbf{a}_\infty\pi_\infty = \mathrm{Clone}(\Pi), \quad \mathbf{A}_\infty\pi_\infty = \mathrm{Clone}^*(\Pi),$$

$$\mathbf{a_3}\pi_\mathbf{1} = \mathrm{Clone}(\pi_{\{1\},\{1\}}), \quad \mathbf{A_3}\pi_\mathbf{1} = \mathrm{Clone}^*(\pi_{\{1\},\{1\}}),$$

$$\mathbf{a_3}\pi_\mathbf{2} = \mathrm{Clone}(\pi_{\{1\},\varnothing}), \quad \mathbf{A_3}\pi_\mathbf{2} = \mathrm{Clone}^*(\pi_{\{1\},\varnothing}).$$

Clones from the class Θ are located in the middle part of the picture. As mentioned above these clones are self-dual with respect to the transposition σ. Clones from the classes Φ and Υ are divided into two symmetric parts, which are dual to each other with respect to the transposition σ. In this paper, we usually refer to the clones from the left-hand part of the picture.

Two vertices M_1 and M_2 of the graph are joined by a solid line and M_1 is located above M_2 iff $M_2 \subset M_1$ and there does not exist a clone M' such that $M_2 \subset M' \subset M_1$. Two vertices M_1 and M_2 are joined by a dotted line and M_1 is located above M_2 iff $M_2 \subset M_1$ and the interval $[M_2, M_1]$ is a countable chain.

In some cases we use a dotted ellipse. Dotted ellipses in the left-hand part of the picture represent the intervals $[\mathbf{a_n}\pi_\infty, \mathbf{a_n}\pi_\mathbf{0}]$ for $n \geq 4$, and the interval $[\mathbf{a}_\infty\pi_\infty, \mathbf{a}_\infty\pi_\mathbf{0}]$. As it follows from Corollary 1.3, the interval $[\mathbf{a_n}\pi_\infty, \mathbf{a_n}\pi_\mathbf{0}]$ is finite for every $n \geq 4$; but these intervals are too complicated to be drawn on a picture. By Theorem 4.29 the interval $[\mathbf{a}_\infty\pi_\infty, \mathbf{a}_\infty\pi_\mathbf{0}]$ has continuum cardinality. In the Hasse diagram of this lattice for every clone from the interval $[\mathbf{a_n}\pi_\infty, \mathbf{a_n}\pi_\mathbf{0}]$ there exists a unique line to the lower layer. Also, for every clone $\mathbf{aP_n}$, where $n \in \mathbb{N}$, there exists a unique line from the interval $[\mathbf{a}_\infty\pi_\infty, \mathbf{a}_\infty\pi_\mathbf{0}]$ to this clone.

To make things more clear we draw the interval $[\mathbf{a_4}\pi_\infty, \mathbf{a_3}\pi_\mathbf{0}]$ on a separate picture (see Figure 3). There you can see the lattice of clones from the class Υ that can be defined by predicates of arity 4. These are all clones from Υ containing the clone $\mathbf{a_4}\pi_\infty$.

A finite set $F \subseteq \Pi$ is placed next to every vertex of the graph in Figure 3. This means that the clone $\mathrm{Clone}(F)$ corresponds to this vertex. Note that the set F is not a downset of Π and hence it does not satisfy the definition of the class Υ. Nevertheless, it is easy to prove for the downset $\downarrow F$ generated by F that $\mathrm{Clone}(F) = \mathrm{Clone}(\downarrow F)$. This follows from Lemma 3.25 and the Galois connection defined in Theorem 2.1.

If a clone is drawn in both figures, then we give in brackets the name of this clone in Figure 2. Note that the lattice in Figure 3 is just the dual of the lattice of downsets of the poset in Figure 1. We hope that now Figure 2 is more clear.

So, pairwise inclusion of clones from Θ and Φ into each other is shown schematically by a graph in Figure 2. The next three theorems describe the inclusion of clones from Υ into clones from Φ and clones from Φ into clones

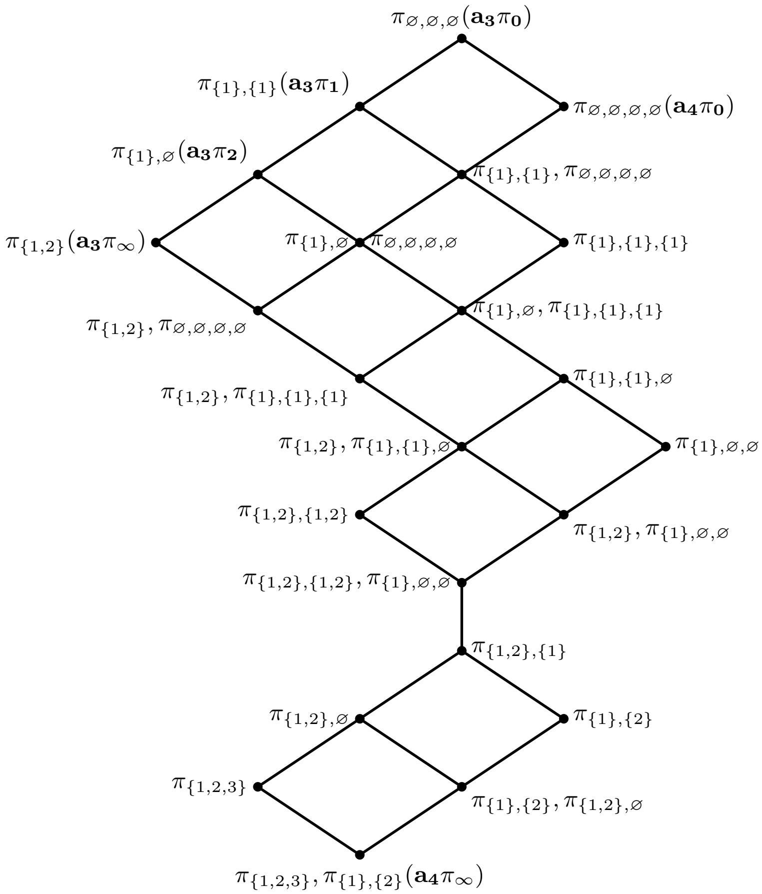

FIGURE 3
Clones from Υ containing $\mathbf{a_4}\pi_\infty$

from Υ.

Theorem 4.8. *Suppose* $t \geq 3$, $F \in \widetilde{\Pi}$, *then* $\mathrm{Clone}(F) \subset \mathbf{a_t N}$ *iff either* $t = 3$ *or* $F \not\subseteq \Pi^{t-1}$.

Theorem 4.12. *Suppose* $F \in \widetilde{\Pi}$, *then* $\mathbf{aP_t} \subset \mathrm{Clone}(F)$ *iff* $F \subseteq \Pi_t$.

Theorem 4.13. *Suppose* $F \in \widetilde{\Pi}$, *then* $\mathbf{aP}_\infty \subset \mathrm{Clone}(F)$.

1.3 Bases for clones

We say that a clone $M \subseteq P_3$ is *finitely generated* if there exists a finite set $M_0 \subseteq M$ such that $M = [M_0]$. A set M_0 is called a *basis* for M if $M = [M_0]$ and for every $M' \subset M_0$ we have $M \neq [M']$.

By $+$ we denote the addition modulo 3.

We use the following notation for functions on E_2: $\overline{x}$, $x \vee y$, $x \wedge y$, $x \oplus y$ are negation, disjunction, conjunction and addition modulo 2 respectively. To reduce formulas in some cases $\wedge$ is omitted. For $n \geq 2$ we put

$$h_n(x_1, \ldots, x_{n+1}) = \bigvee_{i=1}^{n+1} x_1 \ldots x_{i-1} x_{i+1} \ldots x_{n+1},$$

h_n^* is dual to h_n with respect to the transposition on E_2.

We want to partially extend some functions on E_2 to functions on E_3 in a natural way. Let $f \in P_2^n$ with $f(c, c, \ldots, c) = c$ for all $c \in E_2$. Then we define f on tuples $(a_1, \ldots, a_n)$ with $\{a_1, \ldots, a_n\} \subseteq \{d, d+1\}$ for some $d \in E_3$ by

$$f(a_1, \ldots, a_n) = f(a_1 - d, \ldots, a_n - d) + d.$$

For all other tuples we leave the function undefined. Obviously, if f is a binary function on E_2, then the extended f is a completely defined function on E_3. Moreover, it can be checked that the extended f is a self-dual function. By $right(x, y)$ we denote the extension of $x \vee y$, by $left(x, y)$ we denote the extension of $x \wedge y$. That is,

$$right(x, y) = \begin{cases} x, & \text{if } x = y; \\ 1, & \text{if } \{x, y\} = \{0, 1\}; \\ 2, & \text{if } \{x, y\} = \{1, 2\}; \\ 0, & \text{if } \{x, y\} = \{0, 2\}. \end{cases}$$

$$left(x, y) = \begin{cases} x, & \text{if } x = y; \\ 0, & \text{if } \{x, y\} = \{0, 1\}; \\ 1, & \text{if } \{x, y\} = \{1, 2\}; \\ 2, & \text{if } \{x, y\} = \{0, 2\}. \end{cases}$$

These two functions are used widely in the paper because all clones from the classes Φ and Υ contain either $right$ or $left$. Note that these functions are not associative.

Suppose $(a_1, \ldots, a_n) \in E_3^n$, then by $\mathrm{Two}(a_1, \ldots, a_n)$ we denote the set of all $b \in E_3$ that occur in the tuple $(a_1, \ldots, a_n)$ more than once. For example, $\mathrm{Two}(0, 1, 2, 1, 0, 1) = \{0, 1\}$.

To define bases for clones we need to define several functions. In the right-hand side of the following definitions, the functions on E_2 mean the extended functions.

$$r_4(x_1,x_2,x_3,x_4) = \begin{cases} x_1 \vee x_2 \vee x_3, & \text{if } |\{x_1,x_2,x_3\}| \le 2; \\ x_4, & \text{if } |\{x_1,x_2,x_3\}| = 3. \end{cases}$$

$$r_3(x_1,x_2,x_3) = \begin{cases} x_1 \vee x_2, & \text{if } |\{x_1,x_2,x_3\}| \le 2; \\ x_1, & \text{if } |\{x_1,x_2,x_3\}| = 3. \end{cases}$$

$$g_1(x_1,x_2,x_3) = \begin{cases} x_1 \vee x_2, & \text{if } |\{x_1,x_2,x_3\}| \le 2; \\ x_1 \wedge x_2, & \text{if } |\{x_1,x_2,x_3\}| = 3. \end{cases}$$

For $n \ge 2$ we put

$$g_n(x_1,\ldots,x_{n+2}) = \begin{cases} x_1 \vee \ldots \vee x_{n+1}, & \text{if } |\{x_1,\ldots,x_{n+2}\}| \le 2; \\ h_n^*(x_1,\ldots,x_{n+1}), & \text{if } |\{x_1,\ldots,x_{n+1}\}| = 2 \text{ and} \\ & |\{x_1,\ldots,x_{n+1},x_{n+2}\}| = 3; \\ x_1, & \text{otherwise.} \end{cases}$$

$$s_N(x_1,x_2,x_3) = \begin{cases} x_1 \vee x_2, & \text{if } |\{x_1,x_2,x_3\}| \le 2; \\ x_3, & \text{if } |\{x_1,x_2,x_3\}| = 3. \end{cases}$$

$$ps(x,y,z) = \begin{cases} x, & \text{if } |\{x,y,z\}| \le 2; \\ y, & \text{if } |\{x,y,z\}| = 3. \end{cases}$$

$$ps_0(x,y,z) = \begin{cases} x, & \text{if } |\{x,y,z\}| \le 2; \\ x+1, & \text{if } |\{x,y,z\}| = 3. \end{cases}$$

$$plus(x,y,z) = \begin{cases} x \oplus y \oplus z, & \text{if } |\{x,y,z\}| \le 2; \\ x, & \text{if } |\{x,y,z\}| = 3. \end{cases}$$

$$plus_0(x,y,z) = \begin{cases} x \oplus y \oplus z, & \text{if } |\{x,y,z\}| \le 2; \\ x+1, & \text{if } |\{x,y,z\}| = 3. \end{cases}$$

$$m(x,y,z) = \begin{cases} h_2(x,y,z), & \text{if } |\{x,y,z\}| \le 2; \\ x, & \text{if } |\{x,y,z\}| = 3. \end{cases}$$

$$m_0(x,y,z) = \begin{cases} h_2(x,y,z), & \text{if } |\{x,y,z\}| \le 2; \\ x+1, & \text{if } |\{x,y,z\}| = 3. \end{cases}$$

$$f_\pi^\infty(x_1,x_2,x_3)=\begin{cases}x_1\vee x_2x_3, & \text{if } |\{x_1,x_2,x_3\}|\le 2;\\ x_1, & \text{if } |\{x_1,x_2,x_3\}|=3.\end{cases}$$

For $n\ge 3$ put

$$f_\pi^n(x_1,\dots,x_{n+1})=\begin{cases}x_1, & \text{if } \mathrm{Two}(x_1,\dots,x_{n+1})=\{0,1,2\};\\ a\vee b, & \text{if } \mathrm{Two}(x_1,\dots,x_{n+1})=\{a,b\};\\ a, & \text{if } \mathrm{Two}(x_1,\dots,x_{n+1})=\{a\}.\end{cases}$$

$$f_0^\infty(x_1,x_2,x_3)=\begin{cases}x_1\vee x_2\overline{x_3}, & \text{if } |\{x_1,x_2,x_3\}|\le 2;\\ x_1, & \text{if } |\{x_1,x_2,x_3\}|=3.\end{cases}$$

$$f_0^n(x_1,\dots,x_{n+1})=\begin{cases}h_n^*(x_1,\dots,x_{n+1}), & \text{if } |\{x_1,\dots,x_{n+1}\}|\le 2;\\ x_1, & \text{if } |\{x_1,\dots,x_{n+1}\}|=3.\end{cases}$$

$$s_0(x_1,x_2,x_3,x_4)=\begin{cases}x_1\vee x_2x_3, & \text{if } |\{x_1,x_2,x_3,x_4\}|\le 2;\\ x_2, & \text{if } |\{x_1,x_2,x_3,x_4\}|=3 \text{ and } x_2=x_3;\\ x_1, & \text{otherwise.}\end{cases}$$

Theorem 4.16. *The clones of the class Θ have the following bases:*

$$\mathbf{S}=[\{x+1,right\}]=[\{x+1,left\}],$$

$$\mathbf{S_0}=[\{2x+2y,right\}]=[\{2x+2y,left\}],$$

$$\mathbf{SL}=[\{2x+2y,x+1\}]=[\{2x+2y+1\}],\quad \mathbf{1S}=[\{x+1\}],$$

$$\mathbf{SL_0}=[\{2x+2y\}],\quad \mathbf{T}=[\{2x+2y,ps\}],$$

$$\mathbf{C}=[\{plus,right\}]=[\{plus,left\}],$$

$$\mathbf{D}=[\{plus,m_0\}]=[\{plus_0,m\}]=[\{plus,m,ps_0\}],$$

$$\mathbf{M}=[\{right,left\}],\quad \mathbf{DM}=[\{m,ps_0\}]=[\{m_0,ps\}],$$

$$\mathbf{DN}=[\{m_0\}],\quad \mathbf{TD}=[\{m,plus\}],\quad \mathbf{TM}=[\{ps,m\}],$$

$$\mathbf{TN}=[\{m\}],\quad \mathbf{L_2}=[\{plus,ps_0\}]=[\{plus_0\}],$$

$$\mathbf{TL_2}=[\{plus\}],\quad \mathbf{C_2}=[\{ps_0\}],\quad \mathbf{TC_2}=[\{ps\}],\quad \mathbf{J_3}=[\{x\}].$$

As it follows from the definition of the classes Υ and Φ, the clones in the left part of Figure 2 are dual with respect to the transposition σ to the clones in the right part of the figure. That is why we give bases only for clones from the left part.

Theorem 4.17. *The clones of the class Φ have the following bases:*

$$\mathbf{a_2} = [\{f_0^\infty, m\}],\ \ \mathbf{a_2M} = [\{ps, right, m\}],\ \ \mathbf{a_2N} = [\{m, right\}].$$

For $n \geq 3$

$$\mathbf{a_n} = [\{f_0^\infty, f_\pi^n\}] = [\{f_0^\infty, f_0^n\}],$$

$$\mathbf{a_nM} = [\{f_\pi^n, ps\}] = [\{f_0^n\}],\ \ \mathbf{a_nN} = [\{f_\pi^n, s_N\}],$$

$$\mathbf{a_\infty} = [\{f_0^\infty\}],\ \ \mathbf{a_\infty M} = [\{f_\pi^\infty, ps\}],\ \ \mathbf{a_\infty N} = [\{f_\pi^\infty, s_N\}].$$

For $n \geq 1$

$$\mathbf{aP} = [\{right, ps\}],\ \ \mathbf{aPN} = [\{s_N\}],\ \ \mathbf{aP_n} = [\{g_n\}],$$

$$\mathbf{aP_\infty} = [\{r_3\}],\ \ \mathbf{aQ} = [\{r_4\}],\ \ \mathbf{aW} = [\{right\}].$$

Theorem 4.18. *For $n \geq 3$ and $m \geq 1$ the clones of the class Υ have the following bases:*

$$\mathbf{a_\infty}\pi_\infty = [\{f_\pi^\infty\}],\ \ \mathbf{a_n}\pi_\infty = [\{f_\pi^n\}],$$

$$\mathbf{a_\infty}\pi_\mathbf{0} = [\{s_0\}],\ \ \mathbf{a_n}\pi_\mathbf{0} = [\{s_0, f_\pi^n\}],$$

$$\mathrm{Clone}(\Pi_m) = [\{g_m, f_\pi^\infty\}],\ \ \mathrm{Clone}(\Pi_1 \cap \Pi^n) = [\{g_1, f_\pi^n\}].$$

To define bases for all clones from the class Υ we will need the following notions. As mentioned before two predicates ρ_1 and ρ_2 from Π are equivalent if $\rho_1 \lesssim \rho_2$ and $\rho_2 \lesssim \rho_1$. Thus, all predicates from Π are divided into equivalence classes. The set of all equivalence classes we denote by E_Π. By $\widehat{\rho}$ we denote the equivalence class that contains $\rho \in \Pi$. Then, the quasiorder $\lesssim$ generates a partial order on the set E_Π. We write $\widehat{\rho}_1 < \widehat{\rho}_2$, if $\widehat{\rho}_1 \lesssim \widehat{\rho}_2$ and $\widehat{\rho}_1 \neq \widehat{\rho}_2$.

Suppose $F \in \widetilde{\Pi}$. Put

$$\mathrm{Bound}(F) := \{\widehat{\rho} \in E_\Pi \mid \widehat{\rho} \not\subseteq F, \forall \widehat{\sigma} \in E_\Pi(\widehat{\sigma} < \widehat{\rho} \Longrightarrow \widehat{\sigma} \subseteq F)\}.$$

A downset F can be regarded as a subset of E_Π, then $\mathrm{Bound}(F)$ is the set of all minimal elements of the complement of F in E_Π.

A set of pairwise incomparable elements is called an *antichain*. Let B_Π be the set of all antichains of E_Π excluding the one that consists of the bottom element $\widehat{\pi}_{\{\varnothing,\varnothing,\varnothing\}}$ only.

Theorem 4.19. $\mathrm{Bound} : \widetilde{\Pi} \to B_\Pi$ *is a bijective mapping.*

It follows from this theorem and Theorem 4.4 that if we find an infinite antichain in E_Π, then we prove that Υ contains a continuum of clones. As an example, the following set can be considered

$$\{\widehat{\pi}_{A_1,A_2,\ldots,A_m} \mid m \geq 2, \forall i\ A_i = \{1,2,\ldots,m\} \setminus \{i\}\}.$$

Suppose $F \in \widetilde{\Pi}$, $\widehat{\rho} \in \mathrm{Bound}(F)$. Put

$$F_{\widehat{\rho}} = F \cup \bigcup_{\widehat{\delta} \in \mathrm{Bound}(F) \setminus \{\widehat{\rho}\}} \widehat{\delta},$$

$$F_0 = F \cup \bigcup_{\widehat{\delta} \in \mathrm{Bound}(F)} \widehat{\delta}.$$

Theorem 4.20. *Suppose* $M \subset \mathrm{Clone}(F)$, $F \in \widetilde{\Pi} \setminus \{\Pi, \Pi_0, \Pi_1, \Pi_2, \Pi_3, \ldots\}$, *and* $g : \mathrm{Bound}(F) \to M$ *is a bijective mapping such that for every* $\widehat{\rho} \in \mathrm{Bound}(F)$ *we have*

$$g(\widehat{\rho}) \in \mathrm{Clone}(F_{\widehat{\rho}}) \setminus \mathrm{Clone}(F_0).$$

Then M *is a basis for* $\mathrm{Clone}(F)$.

Corollary 4.21. *Suppose* $M \in \Theta \cup \Phi \cup \Upsilon$, *then* M *has a basis.*

Corollary 4.22. *Suppose* $F \in \widetilde{\Pi}$, *then* $\mathrm{Clone}(F)$ *is finitely generated iff* $\mathrm{Bound}(F)$ *is finite.*

Corollary 4.23. *Suppose* $F \in \widetilde{\Pi}$, $|F| < \infty$, *then* $\mathrm{Clone}(F)$ *is finitely generated.*

1.4 Some properties of clones from Θ, Φ, and Υ

The *relation degree* $d(A)$ of a clone $A \subseteq P_3$ is the smallest $h \in \mathbb{N}_0$ such that $A = \mathrm{Pol}(S)$ for some $S \subseteq R_3^h$, that is,

$$d(A) = min\{h \mid \exists Q \subseteq R_3^h : \mathrm{Pol}(Q) = A\}.$$

Put $d(A) = \infty$ if $A \neq \mathrm{Pol}(Q)$ for every finite set $Q \subseteq R_3$.

Theorem 4.25. *Suppose $M \in \Theta \cup \Phi$, then*

$$d(M) = \begin{cases} 2, & \text{if } M \in \{\mathbf{S}, \mathbf{S_0}, \mathbf{T}, \mathbf{C}, \mathbf{M}, \mathbf{D}, \mathbf{DM}, \mathbf{DN}, \mathbf{TD}, \mathbf{TM}, \\ & \mathbf{TN}, \mathbf{1S}, \mathbf{J_3}\}; \\ 3, & \text{if } M \in \{\mathbf{SL}, \mathbf{SL_0}, \mathbf{L_2}, \mathbf{TL_2}, \mathbf{C_2}, \mathbf{TC_2}, \mathbf{aP}, \mathbf{aPN}, \\ & \mathbf{aP_1}, \mathbf{aQ}, \mathbf{aW}, \mathbf{AP}, \mathbf{APN}, \mathbf{AP_1}, \mathbf{AQ}, \mathbf{AW}\}; \\ n, & \text{if } n \geq 2 \text{ and } M \in \{\mathbf{a_n}, \mathbf{a_n M}, \mathbf{a_n N}, \mathbf{A_n}, \mathbf{A_n M}, \mathbf{A_n N}\} \\ n+1, & \text{if } n \geq 2 \text{ and } M \in \{\mathbf{aP_n}, \mathbf{AP_n}\}; \\ \infty, & \text{if } M \in \{\mathbf{a}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}, \mathbf{aP}_\infty, \mathbf{A}_\infty, \mathbf{A}_\infty\mathbf{M}, \\ & \mathbf{A}_\infty\mathbf{N}, \mathbf{AP}_\infty\}; \end{cases}$$

Theorem 4.26. *Suppose $F \in \widetilde{\Pi}$, $F \neq \{\pi_{\varnothing,\varnothing,\varnothing}\}$, then*

$$d(\mathrm{Clone}(F)) = \begin{cases} \max\{m+n \mid \Pi_n^m \cap F \neq \varnothing\}, & \text{if } |F| < \infty; \\ \infty, & \text{otherwise.} \end{cases}$$

$d(\mathrm{Clone}(\{\pi_{\varnothing,\varnothing,\varnothing}\})) = 2.$

It follows from Theorem 4.25 and Theorem 4.26 that our description of clones is optimal. That is, these clones cannot be defined by predicates of smaller arities.

Further, let $\mathbb{L}_3$ be the set of all clones in $\Theta \cup \Phi \cup \Upsilon$. For $F \in \mathbb{L}_3$ we put

$$\mathbb{L}_3^{\uparrow}(F) := \{F' \in \mathbb{L}_3 \mid F \subseteq F'\},$$

$$\mathbb{L}_3^{\downarrow}(F) := \{F' \in \mathbb{L}_3 \mid F' \subseteq F\}.$$

That is, $\mathbb{L}_3^{\uparrow}(F)$ is the principal filter generated by F, and $\mathbb{L}_3^{\downarrow}(F)$ is the principal ideal generated by F.

Theorem 4.30. *Suppose $M \in \Theta \cup \Phi$, then*

$$|\mathbb{L}_3^{\downarrow}(M)| \begin{cases} = \aleph_0, & \text{if } M \in \{\mathbf{aP}, \mathbf{aPN}, \mathbf{aP_1}, \mathbf{aP_2}, \mathbf{aP_3}, \dots, \\ & \mathbf{AP}, \mathbf{APN}, \mathbf{AP_1}, \mathbf{AP_2}, \mathbf{AP_3}, \dots\}; \\ = 2^{\aleph_0}, & \text{if } M \in \{\mathbf{S}, \mathbf{S_0}, \mathbf{C}, \mathbf{M}, \mathbf{a}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}, \\ & \mathbf{A}_\infty, \mathbf{A}_\infty\mathbf{M}, \mathbf{A}_\infty\mathbf{N}\} \\ & \text{or } M \in \bigcup_{n\geq 2}\{\mathbf{a_n}, \mathbf{a_n M}, \mathbf{a_n N}, \mathbf{A_n}, \mathbf{A_n M}, \mathbf{A_n N}\}; \\ < \infty, & \text{otherwise.} \end{cases}$$

Theorem 4.29. *Suppose* $F \in \widetilde{\Pi}$, *then*

$$|\mathbb{L}_3^{\downarrow}(\mathrm{Clone}(F))| = \begin{cases} 2^{\aleph_0}, & \text{if } F \neq \Pi; \\ 5, & \text{if } F = \Pi. \end{cases}$$

Theorem 4.38. *Suppose* $M \in \Theta \cup \Phi$, *then*

$$|\mathbb{L}_3^{\uparrow}(M)| \begin{cases} = \aleph_0, & \text{if } M \in \{\mathbf{C_2}, \mathbf{TC_2}, \mathbf{a}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}, \mathbf{aP}, \mathbf{aPN}, \\ & \mathbf{A}_\infty, \mathbf{A}_\infty\mathbf{M}, \mathbf{A}_\infty\mathbf{N}, \mathbf{AP}, \mathbf{APN}\} \\ & \text{or } M \in \bigcup\limits_{n \geq 1} \{\mathbf{aP_n}, \mathbf{AP_n}\}; \\ = 2^{\aleph_0}, & \text{if } M \in \{\mathbf{J_3}, \mathbf{aP}_\infty, \mathbf{aQ}, \mathbf{aW}, \mathbf{AP}_\infty, \mathbf{AQ}, \mathbf{AW}, \}; \\ < \infty, & \text{otherwise.} \end{cases}$$

Let Π_W be the set of all $\pi_{A_1,\ldots,A_m} \in \Pi$ such that $A_i = A_1 \cup A_2 \cup \ldots \cup A_m$ for some $i \in \{1, 2, \ldots, m\}$. In other words, if $\pi_{A_1,\ldots,A_m} \in \Pi_n^m \cap \Pi_W$, then there exists $i \in \{1, 2, \ldots, m\}$ such that $A_i = \{1, 2, \ldots, n\}$.

Theorem 4.37. *Suppose* $F \in \widetilde{\Pi}$, *then*

$$|\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))| \begin{cases} < \infty, & \text{if } |F| < \infty; \\ = \aleph_0, & \text{if } |F| = \infty, F \subseteq (\Pi_n \cup \Pi_W) \text{ for some } n \in \mathbb{N}; \\ = 2^{\aleph_0}, & \text{otherwise.} \end{cases}$$

Corollary 1.3. $|\mathbb{L}_3^{\uparrow}(\mathbf{a_n}\pi_\infty)| < \infty$ *for every* $n \geq 3$.

It follows from Theorem 4.38 and Corollary 1.3 that $|\mathbb{L}_3^{\uparrow}(\mathbf{aP_m})| = \aleph_0$ for every $m \geq 1$, $|\mathbb{L}_3^{\uparrow}(\mathbf{a_n}\pi_\infty)| < \infty$ for every $n \geq 3$. Roughly speaking, this means that a continuum of clones is located near the vertex $\mathbf{a}_\infty\pi_\infty$ in Figure 2.

2 NECESSARY NOTIONS

2.1 Closure operator for predicates and Galois connection

By $\sigma_k^{=}$ we denote the predicate from R_k given by

$$\sigma_k^{=}(x, y) = 1 \Longleftrightarrow x = y.$$

By $false$ we denote the predicate of arity 0 that takes on value 0, by $true$ we denote the predicate of arity 0 that takes on value 1.

Let us give a short definition of the closure operator [ ] on the set R_k. The reader can find a rigorous definition in the monograph [7]. Suppose $S \subseteq R_k$, then by $[S]$ we denote the set of all predicates $\rho \in R_k$ that can be presented by a formula as follows

$$\rho(x_1, \ldots, x_n) = \exists y_1 \ldots \exists y_l \; \rho_1(z_{1,1}, \ldots, z_{1,n_1}) \wedge \ldots \wedge \rho_s(z_{s,1}, \ldots, z_{s,n_s}),$$

where $l \geq 0$, $\rho_1, \ldots, \rho_s \in S \cup \{false, \sigma_k^=\}$, $z_{i,j} \in \{x_1, \ldots, x_n, y_1, \ldots, y_l\}$. That is a formula over the set $S \cup \{false, \sigma_k^=\}$ from first order predicate logic, which only uses the connective $\wedge$ and existential quantification. The closed subsets $S \subseteq R_k$ with respect to the closure [ ] are called *relational clones*.

Theorem 2.1. *[1, 2, 7] Let $\mathbb{L}(P_k)$ be the set of all clones of P_k, $\mathbb{L}(R_k)$ be the set of all relational clones of R_k. Then the mappings*

$$\mathrm{Inv} : \mathbb{L}(P_k) \longrightarrow \mathbb{L}(R_k),$$

$$\mathrm{Pol} : \mathbb{L}(R_k) \longrightarrow \mathbb{L}(P_k)$$

are mutually inverse bijective mappings, which reverse the partial order $\subseteq$, $i.\,e.$, it holds

$$\forall A, B \in \mathbb{L}(P_k) : A \subseteq B \Rightarrow \mathrm{Inv}(B) \subseteq \mathrm{Inv}(A),$$

$$\forall S, T \in \mathbb{L}(R_k) : S \subseteq T \Rightarrow \mathrm{Pol}(T) \subseteq \mathrm{Pol}(S).$$

So we have a one-to-one correspondence (which is a Galois connection) between clones and relational clones.

2.2 Auxiliary definitions

By $ar(\rho)$ we denote the arity of a predicate ρ. A predicate is called *trivial* if it takes value 1 on every tuple. We say that two predicates ρ_1 and ρ_2 are *equivalent with respect to the set of predicates S* if $\rho_2 \in [S \cup \{\rho_1\}]$ and $\rho_1 \in [S \cup \{\rho_2\}]$.

We say that the i-th variable of a predicate $\rho \in R_k^n$ is *dummy* if for every $a_1, a_2, \ldots, a_n, b \in E_k$ we have

$$\rho(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) = \rho(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

Suppose $\rho \in R_k^n$, $i \leq n$, then we put $\mathrm{VarValues}(\rho, i) = \{\alpha(i) \mid \alpha \in \rho\}$.

We say that $\rho' \in R_3^n$ is obtained from $\rho \in R_3^n$ *by a permutation of variables* if there exists a permutation $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ such that

$$\rho'(x_1, x_2, \ldots, x_n) = \rho(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}).$$

We say that $\rho' \in R_3^n$ is obtained from $\rho \in R_3^n$ *by shifting of variables* if there exist $a_1, a_2, \ldots, a_n \in E_3$ such that

$$\rho'(x_1, x_2, \ldots, x_n) = \rho(x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n).$$

Lemma 2.2. *Suppose $\rho' \in R_3^n$ is obtained from $\rho \in R_3^n$ by shifting of variables, then predicates ρ and ρ' are equivalent with respect to $\{\rho_{+1}\}$.*

Proof. Suppose $\rho'(x_1, x_2, \ldots, x_n) = \rho(x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n)$. Put $\sigma_i(x, y) = \sigma_k^=(x, y)$ if $a_i = 0$; $\sigma_i(x, y) = \rho_{+1}(x, y)$ if $a_i = 1$; $\sigma_i(x, y) = \rho_{+1}(y, x)$ if $a_i = 2$. Then,

$$\rho'(x_1, \ldots, x_n) = \exists y_1 \ldots \exists y_n \ \rho(y_1, \ldots, y_n) \wedge \bigwedge_i \sigma_i(x_i, y_i).$$

□

Suppose $\rho \in R_3$, then by $\mathrm{Shift}(\rho)$ we denote the set of all predicates that can be obtained from ρ by shifting and permutation of variables. For $S \subseteq R_3$ put

$$\mathrm{Shift}(S) = \bigcup_{\rho \in S} \mathrm{Shift}(\rho).$$

Using Lemma 2.2, we get $\mathrm{Shift}(S) \subseteq [S \cup \{\rho_{+1}\}]$ for every S.

Suppose $S \subseteq R_k$, then by $\mathrm{And}(S)$ we denote the set of all $\rho \in R_k$ that can be presented by a formula of the following form:

$$\rho(x_1, \ldots, x_n) = \rho_1(z_{1,1}, \ldots, z_{1,n_1}) \wedge \ldots \wedge \rho_s(z_{s,1}, \ldots, z_{s,n_s}),$$

where $s \geq 0$, $\rho_1, \ldots, \rho_s \in S$, $z_{i,j} \in \{x_1, \ldots, x_n\}$, $z_{i,j} \neq z_{i,l}$ for all $i, j, l, j \neq l$. Here we suppose that ρ is a constant 1 for $s = 0$.

Suppose $\rho \in R_k^n$, $1 \leq i \leq n$. By $\mathrm{Strike}(\rho, i)$ we denote the predicate $\sigma \in R_k^{n-1}$ such that

$$\sigma(x_1, x_2, \ldots, x_{n-1}) = \exists y \ \rho(x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-1}).$$

If $\rho' = \mathrm{Strike}(\rho, i)$, then we say that ρ' is obtained from ρ *by striking the i-th row*. Suppose $\rho \in R_k$, then by $\mathrm{Strike}(\rho)$ we denote the set of all ρ' that can be presented by a formula of the following form:

$$\rho'(x_1, x_2, \ldots, x_n) = \exists y_1 \exists y_2 \ldots \exists y_l \ \rho(z_1, z_2, \ldots, z_m)$$

where $l \geq 0$, $z_1, z_2, \ldots, z_m \in \{x_1, \ldots, x_n, y_1, \ldots, y_l\}$, $z_i \neq z_j$ for $i \neq j$. For $S \subseteq R_3$ we put

$$\mathrm{Strike}(S) = \bigcup_{\rho \in S} \mathrm{Strike}(\rho).$$

Lemma 2.3. *Suppose* $\rho \in R_k$, $c \in E_k$, *and*

$$\rho'(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \rho(x_1, \dots, x_{i-1}, c, x_{i+1}, \dots, x_n).$$

Then $\rho' \in [\{\rho, \{c\}\}]$.

Proof. $\rho'(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) = \exists x_i\ \rho(x_1, \dots, x_n) \wedge (x_i = c)$.

□

Suppose α is a word. Then by $|\alpha|$ we denote the length of α. Suppose $|\alpha| \geq l$, then we put

$$[_l(\alpha) = \alpha(|\alpha| - l + 1) \dots \alpha(|\alpha| - 1)\alpha(|\alpha|),$$

$$]_l(\alpha) = \alpha(1)\alpha(2) \dots \alpha(l).$$

Suppose $s \in \mathbb{N}$, then we put $\alpha^s = \underbrace{\alpha\alpha \dots \alpha}_{s}$.

Suppose $\rho_1, \rho_2 \in R_k^n$; we say that $\rho_1 \leq \rho_2$ if for every $a_1, \dots, a_n \in E_k$

$$\rho_1(a_1, \dots, a_n) \leq \rho_2(a_1, \dots, a_n);$$

we say that $\rho_1 < \rho_2$ if $\rho_1 \leq \rho_2$ and $\rho_1 \neq \rho_2$.

2.3 Essential predicates

A predicate ρ of arity n is called *essential* if there do not exist predicates $\rho_1, \rho_2, \dots, \rho_l$ such that $\mathrm{ar}(\rho_i) < n$ for every $i \in \{1, 2, \dots, l\}$ and $\rho \in \mathrm{And}(\{\rho_1, \rho_2, \dots, \rho_l\})$. We put by definition that $false$ and $true$ are essential predicates. So, $\rho \in R_k$ is called essential if ρ cannot be presented as a conjunction of predicates with arity less than the arity of ρ. The set of all essential predicates of arity n is denoted by $\widetilde{R}_k^n$. Let

$$\widetilde{R}_k = \bigcup_{n \geq 0} \widetilde{R}_k^n.$$

A tuple $(a_1, a_2, \dots, a_n)$ is called *essential for a predicate* $\rho \in R_k^n$ if

$$\rho(a_1, a_2, \dots, a_n) = 0$$

and there exist $b_1, b_2, \dots, b_n \in E_k$ such that for every $i \in \{1, 2, \dots, n\}$

$$\rho(a_1, \dots, a_{i-1}, b_i, a_{i+1}, \dots, a_n) = 1.$$

Let us define the predicate $\widetilde{\rho}$ for every predicate $\rho \in R_k^n$, where $n \geq 1$. Put $\sigma_i = \mathrm{Strike}(\rho, i)$. By $\widetilde{\rho}$ we denote the following predicate:

$$\widetilde{\rho}(x_1, \dots, x_n) = \sigma_1(x_2, \dots, x_n) \wedge \dots \wedge \sigma_n(x_1, \dots, x_{n-1}).$$

Lemma 2.4. *Suppose $\rho \in R_k^n$, where $n \geq 1$. Then the following conditions are equivalent:*

1. *ρ is an essential predicate;*
2. *$\rho \neq \widetilde{\rho}$;*
3. *there exists an essential tuple for ρ.*

Proof. Let $\sigma_i = \mathrm{Strike}(\rho, i)$. We have

$$\widetilde{\rho}(x_1, \ldots, x_n) = \sigma_1(x_2, \ldots, x_n) \wedge \ldots \wedge \sigma_n(x_1, \ldots, x_{n-1}).$$

Let us prove that the first condition implies the second condition, the second implies the third and the third implies the first.

Suppose ρ is essential, then it follows from the definition that $\rho \neq \widetilde{\rho}$.

Suppose $\rho \neq \widetilde{\rho}$. It can be easily checked that $\rho \leq \widetilde{\rho}$. Then there exists $(a_1, \ldots, a_n)$ such that $\widetilde{\rho}(a_1, \ldots, a_n) = 1$, $\rho(a_1, \ldots, a_n) = 0$. By the definition of the predicates $\sigma_1, \ldots, \sigma_n$, for every i there exists $b_i \in E_k$ such that

$$\rho(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) = 1.$$

Hence, the tuple $(a_1, \ldots, a_n)$ is an essential tuple for ρ.

Suppose $(a_1, \ldots, a_n)$ is an essential tuple for ρ. Assume that ρ is not essential. Then there exist $\rho_1, \ldots, \rho_l \in R_k$ such that $\rho \in \mathrm{And}(\{\rho_1, \ldots, \rho_l\})$ and $\mathrm{ar}(\rho_j) < n$ for every j. Without loss of generality it can be assumed that

$$\rho(x_1, \ldots, x_n) = \rho_1(x_1, \ldots, x_n) \wedge \ldots \wedge \rho_l(x_1, \ldots, x_n)$$

and every predicate ρ_j has at least one dummy variable. Since we have $\rho(a_1, \ldots, a_n) = 0$, there exist $j \in \{1, 2, \ldots, l\}$ and $i \in \{1, 2, \ldots, n\}$ such that $\rho_j(a_1, \ldots, a_n) = 0$ and the i-th variable of ρ_j is dummy. Hence, there is no b_i such that $\rho_j(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n) = 1$. Therefore, $(a_1, a_2, \ldots, a_n)$ is not an essential tuple. This contradiction completes the proof.

□

Lemma 2.5. *Suppose $\rho \in R_k$, then $\rho \in \mathrm{And}(\mathrm{Strike}(\rho) \cap \widetilde{R}_k)$.*

Proof. The proof is by induction on the arity of ρ. If $\mathrm{ar}(\rho) = 0$, then ρ is essential and the proof is trivial. If ρ is an essential predicate, then the proof is trivial. Suppose ρ is not essential, $\sigma_i = \mathrm{Strike}(\rho, i)$. Then by Lemma 2.4

$$\rho(x_1, x_2, \ldots, x_n) = \sigma_1(x_2, \ldots, x_n) \wedge \ldots \wedge \sigma_n(x_1, \ldots, x_{n-1}).$$

By the inductive assumption, we have $\sigma_i \in \mathrm{And}(\mathrm{Strike}(\sigma_i) \cap \widetilde{R}_k)$. Hence,

$$\rho \in \mathrm{And}\left(\bigcup_{i=1}^{n} \mathrm{And}\left(\mathrm{Strike}(\sigma_i) \cap \widetilde{R}_k\right)\right) \subseteq \mathrm{And}\left(\bigcup_{i=1}^{n} \mathrm{Strike}(\sigma_i) \cap \widetilde{R}_k\right).$$

Since $\mathrm{Strike}(\sigma_i) \subset \mathrm{Strike}(\rho)$ for every i, we have $\rho \in \mathrm{And}(\mathrm{Strike}(\rho) \cap \widetilde{R}_k)$.

□

Suppose $S \subseteq R_k, n \geq 1$. A predicate $\rho \in R_k^n$ is called *maximal with respect to* S if there exists an essential word (tuple) α for ρ such that the following condition holds:

$$\forall \sigma \in R_k^n \ \ (\sigma > \rho \wedge \sigma(\alpha) = 0) \Rightarrow \sigma \notin [\{\rho\} \cup S].$$

Thus, ρ is a maximal predicate among all predicates $\sigma \in [\{\rho\} \cup S] \cap R_k^n$ such that $\sigma(\alpha) = 0$. The word α is called *a key word* for ρ. By definition we put that predicates $true$ and $false$ are maximal with respect to S for every $S \subseteq R_k$.

Lemma 2.6. *Suppose $\rho \in R_k$, $S \subseteq R_k$, then there exists $W \subseteq [\{\rho\} \cup S]$ such that*

1. *every $\sigma \in W$ is a maximal predicate with respect to S;*
2. $\mathrm{ar}(\sigma) \leq \mathrm{ar}(\rho)$ *for every $\sigma \in W$;*
3. $\rho \in \mathrm{And}(W)$.

Proof. The proof is by induction on the arity of ρ. Let $n = \mathrm{ar}(\rho)$. If $n = 0$, then ρ is maximal with respect to S and the proof is trivial.

Let $\alpha_1, \alpha_2, \ldots, \alpha_l$ be all essential words for ρ. For every $i \in \{1, 2, \ldots, l\}$ let δ_i be a maximal predicate such that $\delta_i \in [\{\rho\} \cup S] \cap R_k^n$, $\delta_i \geq \rho$, and $\delta_i(\alpha_i) = 0$. Obviously, δ_i exists. Let $\sigma_i = \mathrm{Strike}(\rho, i)$. It can be easily checked that we have the following equation

$$\rho(x_1, \ldots, x_n) = \delta_1(x_1, \ldots, x_n) \wedge \ldots \wedge \delta_l(x_1, \ldots, x_n) \wedge \\ \sigma_1(x_2, \ldots, x_n) \wedge \ldots \wedge \sigma_n(x_1, \ldots, x_{n-1}).$$

By the inductive assumption $\sigma_j \in \mathrm{And}(\{\rho_{j,1}, \ldots, \rho_{j,p_j}\})$, where $\rho_{j,i}$ is a maximal predicate with respect to S, $\rho_{j,i} \in [\{\sigma_j\} \cup S] \subseteq [\{\rho\} \cup S]$ and $\mathrm{ar}(\rho_{j,i}) \leq n - 1$. Hence

$$\rho \in \mathrm{And}(\{\delta_1, \ldots, \delta_l, \rho_{1,1}, \ldots, \rho_{1,p_1}, \ldots, \rho_{n,1}, \ldots, \rho_{n,p_n}\}).$$

This completes the proof.

□

2.4 Essential closure

A set $S \subseteq \widetilde{R}_k$ is called *essentially closed* if the following conditions hold:

1. $\sigma_k^=, false \in S$;

2. If ρ is obtained from $\rho_1 \in S$ by a permutation of variables, then $\rho \in S$;

3. If $\rho \in Strike(S) \cap \widetilde{R}_k$, then $\rho \in S$;

4. If $\rho_1 \in S$ and $\rho(x_1, x_2, \ldots, x_n) = \rho_1(x_1, x_1, x_2, \ldots, x_n)$, then either $\rho \notin \widetilde{R}_k$ or $\rho \in S$;

5. If $\rho_1 \in S$ and $\rho(x_1, x_2, \ldots, x_{n-1}) = \exists x \ \rho_1(x, x, x_1, x_2, \ldots, x_{n-1})$, then either $\rho \notin \widetilde{R}_k$ or $\rho \in S$;

6. If $\rho_1, \rho_2 \in S$, $m \le n$, and
$$\rho(x_1, x_2, \ldots, x_n) = \rho_1(x_1, x_2, \ldots, x_n) \wedge \rho_2(x_1, x_2, \ldots, x_m),$$
then either $\rho \notin \widetilde{R}_k$ or $\rho \in S$;

7. If $\rho_1, \rho_2 \in S$, $\mathrm{ar}(\rho_2) = 1$, and
$$\rho(x_1, x_2, \ldots, x_n) = \exists x \ \rho_1(x, x_1, x_2, \ldots, x_n) \wedge \rho_2(x),$$
then either $\rho \notin \widetilde{R}_k$ or $\rho \in S$;

8. If $2 \le l \le k$, $\rho_1, \ldots, \rho_l \in S$, $\mathrm{ar}(\rho_i) = n_i + 1 \ge 2$ for every $i \in \{1, 2, \ldots, l\}$,
$$\rho(x_{1,1}, \ldots, x_{1,n_1}, \ldots, x_{l,1}, \ldots, x_{l,n_l}) = \exists x \ \rho_1(x, x_{1,1}, \ldots, x_{1,n_1}) \wedge \ldots \wedge \rho_l(x, x_{l,1}, \ldots, x_{l,n_l}),$$
where all variables are different; then either $\rho \notin \widetilde{R}_k$ or $\rho \in S$.

Lemma 2.7. *$[Q \cap \widetilde{R}_k] = Q$ for every relational clone $Q \subseteq R_k$.*

Proof. The inclusion $[Q \cap \widetilde{R}_k] \subseteq Q$ is trivial. Let us prove the inclusion $[Q \cap \widetilde{R}_k] \supseteq Q$. Suppose $\rho \in Q$, then by Lemma 2.5, it follows that $\rho \in \mathrm{And}(\mathrm{Strike}(\rho) \cap \widetilde{R}_k)$. Since $\mathrm{Strike}(\rho) \subseteq [\{\rho\}]$ and $\mathrm{And}(T) \subseteq [T]$ for every $T \subseteq R_k$, we get $\rho \in [[\{\rho\}] \cap \widetilde{R}_k] \subseteq [Q \cap \widetilde{R}_k]$. This concludes the proof. □

The following theorem is proved at the end of this section.

Theorem 2.8. *A set $S \subseteq \widetilde{R}_k$ is essentially closed iff $[S] \cap \widetilde{R}_k = S$.*

It follows from Lemma 2.7 that an arbitrary relational clone Q can be uniquely determined by the set $Q \cap \widetilde{R}_k$. Moreover, it follows from Theorem 2.8 that $Q \cap \widetilde{R}_k$ is an essentially closed set of predicates. So, we have a one-to-one correspondence between relational clones and essentially closed sets of essential predicates. Thus, to describe all clones in three-valued logic, it is sufficient to describe all essentially closed sets of essential predicates.

Let us consider a simple example. Suppose $\rho \in R_k^2$ defines a linear order on the set E_k. Let $\rho'(x,y) = \rho(y,x)$. It is easy to check that the set $\{\sigma_k^=, false, true, \rho, \rho'\}$ is essentially closed. Hence, $\mathrm{Pol}(\rho)$ is a maximal (or precomplete) clone in P_k. Note that we prove this without using functions at all.

The following lemmas will be used in the proof of Theorem 2.8.

Lemma 2.9. *Suppose $\rho, \rho_1, \dots, \rho_l \in R_k$, $\mathrm{ar}(\rho_i) = n_i + 1 \geq 2$, $l > k$, and*

$$\rho(x_{1,1}, \dots, x_{1,n_1}, x_{2,1}, \dots, x_{2,n_2}, \dots, x_{l,1}, \dots, x_{l,n_l}) = \\ \exists y\ \rho_1(y, x_{1,1}, \dots, x_{1,n_1}) \wedge \dots \wedge \rho_l(y, x_{l,1}, \dots, x_{l,n_l}).$$

Then ρ is not an essential predicate.

Proof. Assume the converse. By Lemma 2.4, there exists an essential tuple γ for ρ. Suppose $\gamma = \alpha_1\alpha_2\dots\alpha_l$ where $\alpha_i \in E_k^{n_i}$ for every $i \in \{1, 2, \dots, m\}$. Put

$$C_i = \{c \in E_k \mid \rho_i(c\alpha_i) = 1\}.$$

Since γ is an essential tuple, we have

$$C_1 \cap C_2 \cap \dots \cap C_l = \varnothing,$$

$$D_j := \bigcap_{i \neq j} C_i \neq \varnothing.$$

Hence, $D_i \cap D_j = \varnothing$ for every i, j, $i \neq j$. Since $D_i \subseteq E_k$ for every i, we have $l \leq k$. This concludes the proof.

□

Lemma 2.10. *Suppose S is an essentially closed set of predicates, $\rho_0 \in S$, $\rho \in \mathrm{Strike}(\rho_0)$. Then $\rho \in \mathrm{And}(S)$.*

Proof. By Lemma 2.5 and using item 3 of the definition, we obtain

$$\rho \in \mathrm{And}(\mathrm{Strike}(\rho) \cap \widetilde{R}_k) \subseteq \mathrm{And}(\mathrm{Strike}(\rho_0) \cap \widetilde{R}_k) \subseteq \mathrm{And}(S).$$

□

Lemma 2.11. *Suppose $\rho \in R_k^n$, $\sigma_i = \mathrm{Strike}(\rho, i)$, predicates ρ and σ_1 are not essential. Then*

$$\rho(x_1, \ldots, x_n) = \sigma_2(x_1, x_3, \ldots, x_n) \wedge \ldots \wedge \sigma_n(x_1, \ldots, x_{n-1}).$$

Proof. Let $\sigma_{1,i} = \mathrm{Strike}(\sigma_1, i-1)$. Using Lemma 2.4 for ρ and σ_1 we get

$$\begin{aligned}
&\rho(x_1, \ldots, x_n) = \\
&\quad \sigma_1(x_2, \ldots, x_n) \wedge \sigma_2(x_1, x_3, \ldots, x_n) \wedge \ldots \wedge \sigma_n(x_1, \ldots, x_{n-1}) = \\
&\bigwedge_{i=2}^{n}(\sigma_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \wedge \sigma_{1,i}(x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)).
\end{aligned}$$

Hence the following equation completes the proof.

$$\sigma_i(x_1, x_2, \ldots, x_{n-1}) \wedge \sigma_{1,i}(x_2, \ldots, x_{n-1}) = \sigma_i(x_1, x_2, \ldots, x_{n-1}).$$

□

Lemma 2.12. *Suppose S is an essentially closed set of predicates, $\rho_0 \in \mathrm{And}(S)$, and*

$$\rho(x_1, x_2, \ldots, x_n) = \rho_0(x_1, x_1, x_2, \ldots, x_n).$$

Then $\rho \in \mathrm{And}(S)$.

Proof. The proof is by induction on $\mathrm{ar}(\rho_0)$. By the condition, ρ_0 can be presented as a conjunction of predicates $\delta_1, \ldots, \delta_s \in S$. Hence, we just need to prove that if we identify two variables in δ_i we obtain a predicate from $\mathrm{And}(S)$. Therefore, without loss of generality we can assume that $\rho_0 \in S$.

If ρ is essential, then the proof follows from item 4 of the definition of an essentially closed set. Suppose ρ is not essential. If $\mathrm{ar}(\rho) = 1$, then obviously $\rho \in \mathrm{And}(S)$. Suppose $\mathrm{ar}(\rho) \geq 2$. By Lemma 2.4 we have $\rho = \widetilde{\rho}$. Let $\sigma_j = \mathrm{Strike}(\rho, j)$. Let us show that $\sigma_j \in \mathrm{And}(S)$ for every $j \in \{2, 3, \ldots, \mathrm{ar}(\rho)\}$. Let $\epsilon_j = \mathrm{Strike}(\rho_0, j+1)$. By Lemma 2.10 we have $\epsilon_j \in \mathrm{And}(S)$. By the inductive assumption we get $\sigma_j \in \mathrm{And}(S)$ for every $j \in \{2, 3, \ldots, \mathrm{ar}(\rho)\}$.

Assume that σ_1 is essential. It follows from item 5 of the definition that $\sigma_1 \in S$. Hence $\rho \in \mathrm{And}(\{\sigma_1, \sigma_2, \ldots, \sigma_{\mathrm{ar}(\rho)}\}) \subseteq \mathrm{And}(S)$.

Suppose σ_1 is not essential. Then using Lemma 2.11 we obtain that $\rho \in \mathrm{And}(\{\sigma_2, \sigma_3, \ldots, \sigma_{\mathrm{ar}(\rho)}\}) \subseteq \mathrm{And}(S)$. This completes the proof.

□

Suppose $S \subseteq \widetilde{R}_k$. By $\mathrm{AF}(S)$ we denote the set of all formulas of the following form:

$$\rho_1(z_{1,1}, \ldots, z_{1,n_1}) \wedge \ldots \wedge \rho_s(z_{s,1}, \ldots, z_{s,n_s}),$$

where $\rho_1, \ldots, \rho_s \in S$, $z_{i,j} \neq z_{i,l}$ for all $i, j, l, j \neq l$.

By Seq we denote the set of all infinite sequences $(a_0, a_1, a_2, \ldots)$ such that $a_i \in \mathbb{N}_0$ for every $i \in \mathbb{N}_0$, and there exists $j \in \mathbb{N}_0$ such that $a_i = 0$ for every $i \geq j$. Let us define a mapping $\varphi : \mathrm{AF}(S) \to \mathrm{Seq}$. Put $\varphi(\Phi) = (a_0, a_1, a_2, \ldots)$, where a_i is the number of predicates of arity i in the formula Φ. Let us define a linear order on the set Seq. We say that $(a_0, a_1, a_2, \ldots) < (b_0, b_1, b_2, \ldots)$ if there exists $m \in \mathbb{N}_0$ such that $a_m < b_m$ and $a_i = b_i$ for every $i > m$.

Lemma 2.13. *Suppose $\varnothing \neq W \subseteq \mathrm{Seq}$. Then there exists a minimal element in W.*

Proof. Consider $\gamma \in W$. Suppose that $\gamma(i) = 0$ for every $i > n$. Let $W_n = \{\alpha \in W \mid \forall i > n\ (\alpha(i) = 0)\}$. For $i \in \{0, 1, 2, \ldots, n\}$ we put

$$b_i = \min\{\alpha(i) \mid \alpha \in W_i\},$$

$$W_{i-1} = \{\alpha \in W_i \mid \alpha(i) = b_i\}.$$

Obviously, W_i is not empty for every $i \in \{0, 1, 2, \ldots, n\}$. Therefore, the sequence $(b_0, b_1, b_2, \ldots, b_n, 0, 0, 0, \ldots)$ is a minimal element in W.

□

Suppose $S \subseteq \widetilde{R}_k$, $\sigma \in \mathrm{And}(S) \cap R_k^n$. By $\mathrm{AF}(\sigma, S, x_1, \ldots, x_n)$ we denote the set of all formulas $\Phi \in \mathrm{AF}(S)$ such that $\sigma(x_1, \ldots, x_n) = \Phi(x_1, \ldots, x_n)$.

Lemma 2.14. *Suppose $S \subseteq \widetilde{R}_k$ is essentially closed, $\rho_0 \in \mathrm{And}(S)$, and*

$$\rho(x_1, \ldots, x_n) = \exists x\ \rho_0(x, x_1, \ldots, x_n).$$

Then $\rho \in \mathrm{And}(S)$.

Proof. Assume the converse. Let $\Psi_0 \in \mathrm{AF}(S)$ be a formula with the minimal value of $\varphi(\Psi)$ such that $\exists x\ \Psi_0$ realizes a predicate $\rho \notin \mathrm{And}(S)$. Let us rename some variables in Ψ_0 such that every variable except x in the obtained formula occurs just once. Note that we remain all occurrences of the variable x in Ψ_0. We denote the obtained formula by Ψ. Suppose $\exists x\ \Psi$ realizes a predicate ρ'. If $\rho' \in \mathrm{And}(S)$, then using Lemma 2.12 we obtain that $\rho \in \mathrm{And}(S)$, which contradicts the assumption.

Assume that $\rho' \notin \mathrm{And}(S)$. Obviously $\varphi(\Psi) = \varphi(\Psi_0)$. Let

$$\Psi = \Xi_1 \wedge \Xi_2 \wedge \ldots \wedge \Xi_r.$$

Without loss of generality it can be assumed that there exists $m \le r$ such that Ξ_i contains x iff $i \le m$. It can be assumed that $\Xi_i = \rho_i(x, x_{i,1}, \ldots, x_{i,n_i})$ for every $i \le m$.

Assume that $m < r$. Let $\Psi_1 = \Xi_1 \wedge \ldots \wedge \Xi_m$. Let $\exists x\ \Psi_1$ realize a predicate δ. Obviously, $\varphi(\Psi_1) < \varphi(\Psi)$. By the assumption about the minimality of $\varphi(\Psi)$ we get $\delta \in \mathrm{And}(S)$. Hence, $\rho' \in \mathrm{And}(S)$, which contradicts the assumption.

Thus, we can assume that $m = r$.

We have three cases. First case, $m = 1$. By Lemma 2.10 the formula $\exists x\ \Xi_1$ realizes a predicate from $\mathrm{And}(S)$. Hence $\rho' \in \mathrm{And}(S)$.

Second case, $\mathrm{ar}(\rho_i) = 1$ for some $i \in \{1, 2, \ldots, m\}$ and $m > 2$. Without loss of generality it can be assumed that $i = 1$. For $j \in \{2, 3, \ldots, m\}$ we put

$$\rho'_j(x, x_{j,1}, \ldots, x_{j,n_j}) = \rho_1(x) \wedge \rho_j(x, x_{j,1}, \ldots, x_{j,n_j}).$$

Assume that ρ'_j is essential for every j. Then it follows from item 6 of the definition that $\rho'_j \in S$ for every j. We put $\Xi'_j = \rho'_j(x, x_{j,1}, \ldots, x_{j,n_j})$. Let

$$\Psi_2 = \Xi'_2 \wedge \Xi'_3 \wedge \ldots \wedge \Xi'_m.$$

Obviously $\exists x\ \Psi_2$ realizes the predicate ρ' and $\varphi(\Psi_2) < \varphi(\Psi)$. This contradicts the assumption about the minimality of $\varphi(\Psi)$.

Assume that ρ'_j is not essential for some $j \in \{2, 3, \ldots, m\}$. Let $\sigma_i = \mathrm{Strike}(\rho'_j, i)$. By the assumption about the minimality of $\varphi(\Psi)$ we have $\sigma_i \in \mathrm{And}(S)$ for every $i \in \{1, 2, \ldots, n_j + 1\}$. Let for $i \in \{1, 2, \ldots, n_j\}$

$$\Theta_0 \in \mathrm{AF}(\sigma_1, S, x_{j,1}, \ldots, x_{j,n_j}),$$

$$\Theta_i \in \mathrm{AF}(\sigma_{i+1}, S, x, x_{j,1}, \ldots, x_{j,i-1}, x_{j,i+1}, \ldots, x_{j,n_j}).$$

By Lemma 2.4, the formula $\Theta_0 \wedge \Theta_1 \wedge \ldots \wedge \Theta_{n_j}$ realizes ρ'_j. Let Ψ_3 be obtained from Ψ by replacing Ξ_j by $\Theta_0 \wedge \Theta_1 \wedge \ldots \wedge \Theta_{n_j}$. Obviously $\exists x\ \Psi_3$ realizes the predicate ρ' and $\varphi(\Psi_3) < \varphi(\Psi)$. This contradicts the assumption about the minimality of $\varphi(\Psi)$.

Third case, $m = 2$ or $\mathrm{ar}(\rho_i) > 1$ for every $i \in \{1, 2, \ldots, m\}$. If ρ' is essential then using Lemma 2.9 we obtain $m \le k$. Hence, it follows from item 7 and item 8 of the definition that $\rho' \in S$. This contradicts the assumption.

Assume that ρ' is not essential. Let $\sigma_i = \mathrm{Strike}(\rho', i)$. Let us prove that $\sigma_i \in \mathrm{And}(S)$ for every i. Without loss of generality it can be assumed that $i = 1$. By Lemma 2.10 the formula $\exists x_{1,1}\ \Xi_1$ realizes a predicate from $\mathrm{And}(S)$. Denote this predicate by σ'. Let $\Omega \in \mathrm{AF}(\sigma', S, x, x_{1,2}, \ldots, x_{1,n_1})$. Then we have

$$\exists x_{1,1}\ \exists x\ \Xi_1 \wedge \Xi_2 \wedge \ldots \wedge \Xi_m = \\ \exists x\ (\exists x_{1,1}\ \Xi_1) \wedge \Xi_2 \wedge \ldots \wedge \Xi_m = \exists x\ \Omega \wedge \Xi_2 \wedge \ldots \wedge \Xi_m.$$

Let $\Psi_4 = \Omega \wedge \Xi_2 \wedge \ldots \wedge \Xi_m$. It can be easily checked that $\varphi(\Psi_4) < \varphi(\Psi)$ and $\exists x\ \Psi_4$ realizes the predicate σ_1. Because of the minimality of $\varphi(\Psi)$ we get $\sigma_1 \in \mathrm{And}(S)$.

Hence, $\sigma_i \in \mathrm{And}(S)$ for every i. By Lemma 2.4 we get $\rho' \in \mathrm{And}(S)$. This contradiction completes the proof.

□

Lemma 2.15. *Suppose S is an essentially closed set, then* $\mathrm{And}(S)$ *is a closed set of predicates.*

Proof. This lemma can be easily proved by combining the definition of essentially closed set, Lemma 2.12 and Lemma 2.14.

□

Proof of Theorem 2.8. If $[S] \cap \widetilde{R}_k = S$ then it is easy to show that S is essentially closed.

Suppose S is an essentially closed set of predicates. By Lemma 2.15 we obtain that $[\mathrm{And}(S)] = \mathrm{And}(S)$. We have

$$\mathrm{And}(S) \subseteq [S] \subseteq [\mathrm{And}(S)] = \mathrm{And}(S).$$

Therefore, $\mathrm{And}(S) = [S]$. Now we must only prove that $\mathrm{And}(S) \cap \widetilde{R}_k \subseteq S$.

Consider $\rho \in \mathrm{And}(S) \cap \widetilde{R}_k$. Suppose $(a_1, \ldots, a_n)$ is an essential tuple for ρ, ρ is presented as a conjunction of predicates $\rho_0, \rho_1, \ldots, \rho_l \in S$. At least one of these predicates takes on value 0 on the tuple $(a_1, \ldots, a_n)$. Hence, this tuple is essential for this predicate. Without loss of generality it can be assumed that $(a_1, \ldots, a_n)$ is an essential tuple for ρ_0. Therefore,

$$\rho(x_1, x_2, \ldots, x_n) = \\ \rho_0(x_1, x_2, \ldots, x_n) \wedge \rho_1(x_{1,1}, \ldots, x_{1,n_1}) \wedge \ldots \wedge \rho_l(x_{l,1}, \ldots, x_{l,n_l}).$$

For $i \in \{1, \ldots, l\}$ we put

$$\begin{aligned} \rho'_0(x_1, x_2, \ldots, x_n) &= \rho_0(x_1, x_2, \ldots, x_n), \\ \rho'_i(x_1, x_2, \ldots, x_n) &= \rho'_{i-1}(x_1, x_2, \ldots, x_n) \wedge \rho_i(x_{i,1}, \ldots, x_{i,n_i}). \end{aligned}$$

It can be easily checked that $(a_1, \ldots, a_n)$ is an essential tuple for every predicate ρ'_i. Hence, ρ'_i is essential for every i. It follows from item 6 of the definition that $\rho'_i \in S$ for every $i \in \{1, 2, \ldots, l\}$. Obviously, ρ'_l is equal to ρ. Therefore, $\rho \in S$ and $\mathrm{And}(S) \cap \widetilde{R}_k \subseteq S$. This completes the proof.

3 CLONES FROM CLASSES Φ AND Υ.

3.1 General properties of predicates from $\mathrm{Inv}(right)$

Let

$$B_0 = \{false, true, \rho_{+1}, \sigma_3^{=}, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\},$$

$$B_1 = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \rho_{\leq}, \rho_{\vee,2}, \rho_N, \rho_W \right\}.$$

By B_2 we denote the set of all predicates $\rho \in \mathrm{Inv}(right)$ such that for every $i \in \{1, 2, \ldots, \mathrm{ar}(\rho)\}$ we have $\mathrm{VarValues}(\rho, i) \subseteq \{0, 1\}$.

Let $Main$ be the set of all predicates $\rho \in R_3$ such that the following conditions hold for some $m \in \{1, \ldots, \mathrm{ar}(\rho)\}$:

1. $\mathrm{VarValues}(\rho, i) \subseteq \{0, 1\}$ for every $i \in \{1, 2, \ldots, m\}$.
2. For every $a_{m+1}, \ldots, a_{\mathrm{ar}(\rho)} \in E_3$ we have

$$\rho(1, \ldots, 1, a_{m+1}, \ldots, a_{\mathrm{ar}(\rho)}) = 1.$$

So, $\mathrm{VarValues}(\rho, i) \subseteq \{0, 1\}$ for every $i \leq m$, $\mathrm{VarValues}(\rho, i) = \{0, 1, 2\}$ for every $i > m$.

Lemma 3.1. $B_0 \subseteq [\{\rho_{+1}, \{0, 1\}\}]$.

Proof. By Lemma 2.2, we have

$$\mathrm{Shift}(\{0, 1\}) = \{\{0, 1\}, \{1, 2\}, \{0, 2\}\} \subseteq [\{\rho_{+1}, \{0, 1\}\}].$$

Also $\{0, 1\} \cap \{1, 2\} = \{1\}$, hence

$$\mathrm{Shift}(\{1\}) = \{\{0\}, \{1\}, \{2\}\} \subseteq [\{\rho_{+1}, \{0, 1\}\}].$$

□

Lemma 3.2. *Suppose $D \subset E_3^m$, $D \neq \varnothing$, then there exist $i \in \{1, 2, \ldots, m\}$, $\alpha \in D, \beta \in E_3^m \setminus D$ such that $\beta(j) = \alpha(j)$ for every $j \neq i$, $\beta(i) = \alpha(i) + 1$.*

Proof. Let $\rho \in R_3^m$ be defined by the following condition

$$\rho(\alpha) = 1 \Longleftrightarrow \alpha \in D.$$

Since ρ takes on value 0 and value 1, we see that at least one of the variables in ρ is not dummy.

Hence there exist $i \in \{1, 2, \ldots, m\}$, $a_1, \ldots, a_m \in E_3$ such that for $\rho'(x) = \rho(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m)$ we have $\rho' \neq E_3$. Thus there is some $c \in E_3$ such that $\rho'(c) = 1$ and $\rho'(c+1) = 0$. We put

$$\alpha = a_1 \ldots a_{i-1} c a_{i+1} \ldots a_m,\ \beta = a_1 \ldots a_{i-1}(c+1)a_{i+1} \ldots a_m.$$

□

Lemma 3.3. *Suppose $\rho \in \mathrm{Inv}(right)$, $\mathrm{ar}(\rho) \geq 1$, $\mathrm{VarValues}(\rho, i) = E_3$ for every $i \in \{1, 2, \ldots, \mathrm{ar}(\rho)\}$. Then $\rho \in \mathrm{And}(\{\rho_{+1}, \sigma_3^{=}\})$.*

Proof. The proof is by induction on the arity of ρ. If $\mathrm{ar}(\rho) = 1$, then the proof is trivial.

Let $n = \mathrm{ar}(\rho)$. Let $\sigma_i = \mathrm{Strike}(\rho, i)$ for $i \in \{1, 2, \ldots, n\}$. By the inductive assumption, $\sigma_i \in \mathrm{And}(\{\rho_{+1}, \sigma_3^{=}\})$.

We have two cases. First case, σ_i is not trivial for some i. Without loss of generality it can be assumed that there exists $\rho_0 \in \{\rho_{+1}, \sigma_3^{=}\}$ such that $\rho_0(\alpha(1), \alpha(n)) = 1$ for every $\alpha \in \rho$. Then we can easily show that

$$\rho(x_1, \ldots, x_n) = \sigma_n(x_1, \ldots, x_{n-1}) \wedge \rho_0(x_1, x_n) \in \mathrm{And}(\{\rho_{+1}, \sigma_3^{=}\}),$$

which completes this case.

Second case, σ_i is trivial for every $i \in \{1, 2, \ldots, n\}$. Suppose $c \in E_3$, then by T_c we denote the set of all $\alpha \in E_3^{n-1}$ such that $\rho(\alpha c) = 1$. Hence $T_0 \cup T_1 \cup T_2 = E_3^{n-1}$.

We have two subcases. First subcase, $T_0 \cap T_1 \neq \varnothing$ (cases $T_1 \cap T_2 \neq \varnothing$ and $T_0 \cap T_2 \neq \varnothing$ are considered in the same way). Let $\alpha \in T_0 \cap T_1$, $\beta \in T_2$. Since $right$ preserves ρ, we have

$$right(\alpha 1, \beta 2) = right(\alpha, \beta)2 \Rightarrow right(\alpha, \beta) \in T_2,$$

$$right(\alpha 0, right(\alpha 1, \beta 2)) = right(\alpha, \beta)0 \Rightarrow right(\alpha, \beta) \in T_0,$$

$$right(\alpha 1, right(\alpha 0, \beta 2)) = right(\alpha, \beta)1 \Rightarrow right(\alpha, \beta) \in T_1.$$

Hence $T_0 \cap T_1 \cap T_2 \neq \varnothing$. If ρ is trivial, then there is nothing to prove. If ρ is not trivial, then $T_0 \cap T_1 \cap T_2 \neq E_3^{n-1}$ and by Lemma 3.2 there exist $\gamma, \delta \in E_3^{n-1}$, such that

$$\gamma \in T_0 \cap T_1 \cap T_2,\ \delta \notin T_0 \cap T_1 \cap T_2,\ right(\gamma, \delta) = \delta.$$

Since σ_n is trivial, there exists $c \in E_3$ such that $\delta \in T_c$. Since $right$ preserves ρ, we have

$$right(\delta c, \gamma(c+1)) = \delta(c+1) \in \rho,$$

$$right(\delta(c+1), \gamma(c+2)) = \delta(c+2) \in \rho.$$

Hence $\delta \in T_0 \cap T_1 \cap T_2$. This contradiction completes this case.

Second subcase, for every $\alpha \in E_3^{n-1}$ there exists a unique $c \in E_3$ such that $\alpha \in T_c$. Obviously, $T_0 \neq \varnothing$ and $T_0 \neq E_3^{n-1}$. Then by Lemma 3.2, there exist $\alpha, \beta \in E_3^{n-1}$ and $i \in \{1, 2, \dots, n-1\}$ such that $\alpha \in T_0$, $\beta \notin T_0$, $\beta(j) = \alpha(j)$ for every $j \neq i$, $\beta(i) = \alpha(i) + 1$. Without loss of generality it can be assumed that $i = 1$.

So, we have $d \in E_3$ and $\gamma \in E_3^{n-2}$ such that $d\gamma \in T_0$, $(d+1)\gamma \notin T_0$. Assume that $(d+1)\gamma \in T_2$, then

$$right(d\gamma 0, (d+1)\gamma 2) = (d+1)\gamma 0 \ \Rightarrow\ (d+1)\gamma \in T_0.$$

This contradiction proves that $(d+1)\gamma \in T_1$.

If $(d+2)\gamma \in T_0$, then

$$right((d+2)\gamma 0, (d+1)\gamma 1) = (d+2)\gamma 1 \ \Rightarrow\ (d+2)\gamma \in T_1.$$

Then $T_0 \cap T_1 \neq \varnothing$, which contradicts the assumption. If $(d+2)\gamma \in T_1$, then

$$right((d+2)\gamma 1, d\gamma 0) = d\gamma 1 \ \Rightarrow\ d\gamma \in T_1$$

Then $T_0 \cap T_1 \neq \varnothing$, which proves that $(d+2)\gamma \in T_2$.

Let $G = \{\delta \in E_3^{n-2} \mid d\delta \in T_0 \wedge (d+1)\delta \in T_1 \wedge (d+2)\delta \in T_2\}$. Since $\gamma \in G$, we see that $G \neq \varnothing$. Assume that $G \neq E_3^{n-2}$, then by Lemma 3.2 there exist $\alpha, \beta \in E_3^{n-2}$ such that

$$\alpha \in G,\ \beta \notin G,\ right(\alpha, \beta) = \beta.$$

Without loss of generality it can be assumed that $d\beta \notin T_0$.

Let us consider two cases.

Suppose $d\beta \in T_1$, then $right(d\beta 1, (d+2)\alpha 2) = d\beta 2 \in \rho$ and $d\beta \in T_2$. Then $T_1 \cap T_2 \neq \varnothing$, and this contradicts the assumption.

Suppose $d\beta \in T_2$, then $right(d\beta 2, d\alpha 0) = d\beta 0 \in \rho$ and $d\beta \in T_0$. Then $T_0 \cap T_2 \neq \varnothing$, and this contradicts the assumption.

Hence, $G = E_3^{n-2}$ and $\rho(x_1, \ldots, x_n) = 1 \Leftrightarrow (x_1 = x_n + d)$. Therefore, $\rho \in \mathrm{And}(\{\rho_{+1}, \sigma_3^{=}\})$.

□

Lemma 3.4. *Suppose* $\rho \in \mathrm{Inv}(right) \cap \widetilde{R}_3$, *then* $\rho \in \mathrm{Shift}(B_0 \cup Main)$.

Proof. If $\mathrm{ar}(\rho) \leq 1$ then $\rho \in B_0$ and there is nothing to prove. Suppose $\mathrm{VarValues}(\rho, i) = E_3$ for every $i \in \{1, 2, \ldots, \mathrm{ar}(\rho)\}$, then by Lemma 3.3 we have $\rho \in \mathrm{And}(\{\rho_{+1}, \sigma_3^{=}\})$. Since ρ is essential we get $\rho \in \mathrm{Shift}(B_0)$. This case is finished.

Suppose $\mathrm{VarValues}(\rho, i) \neq E_3$ for some $i \in \{1, 2, \ldots, \mathrm{ar}(\rho)\}$. Let

$$m = |\{i \mid \mathrm{VarValues}(\rho, i) \neq E_3\}|.$$

It can be shown that there exists $\rho' \in \mathrm{Shift}(\rho)$ such that $\mathrm{VarValues}(\rho', i) \subseteq \{0, 1\}$ for $i \leq m$ and $\mathrm{VarValues}(\rho', i) = E_3$ for $i > m$. Since ρ' is essential, we see that $\mathrm{VarValues}(\rho', i) = \{0, 1\}$ for every $i \leq m$. Let $n = \mathrm{ar}(\rho) - m$.

To complete the proof we need to show that $\rho'(1, \ldots, 1, b_1, \ldots, b_n) = 1$ for every $b_1, b_2, \ldots, b_n \in E_3$. Put

$$\rho_0(y_1, \ldots, y_n) = \exists x_1 \ldots \exists x_m \ \ \rho'(x_1, \ldots, x_m, y_1, \ldots, y_n).$$

By Lemma 3.3 we obtain that $\rho_0 \in \mathrm{And}(\{\rho_{+1}, \sigma_3^{=}\})$. If ρ_0 is not trivial then it can be easily checked that there is no an essential tuple for ρ', hence ρ' is not essential. This contradiction proves that ρ_0 is trivial.

For every $i \leq m$, since $1 \in \mathrm{VarValues}(\rho', i)$ there exists $\delta_i \in \rho'$ such that $\delta_i(i) = 1$. Let

$$\gamma = right(\delta_1, right(\delta_2, right(\ldots(right(\delta_{m-1}, \delta_m))\ldots))).$$

Obviously $\gamma(i) = 1$ for every $i \in \{1, 2, \ldots, m\}$. Since $right \in \mathrm{Pol}(\rho')$, we see that $\gamma \in \rho'$.

Let $\gamma' = [_n(\gamma)$. Let $G = \{\delta \in E_3^n \mid \rho'(1^m \delta) = 1\}$. Obviously, $\gamma' \in G$ and $G \neq \varnothing$. Assume that $G \neq E_3^n$. By Lemma 3.2 there exist $\alpha, \beta \in E_3^n$ such that

$$\alpha \in G, \ \beta \notin G, \ right(\alpha, \beta) = \beta.$$

Since ρ_0 is trivial, there exists β_0 such that $\beta_0\beta \in \rho'$. Since $right \in \mathrm{Pol}(\rho')$, we obtain

$$right(\beta_0\beta, 1^m\alpha) = 1^m\beta \in \rho'.$$

This contradiction proves that $G = E_3^n$. This completes the proof.

□

3.2 Clones $\mathbf{aQ}$ and $\mathbf{aW}$.

Lemma 3.5. $B_0 \cup B_1 \subseteq [\{\rho_W, \rho_{+1}\}]$.

Remark. Note that $[\{\rho_W, \rho_{+1}\}] = \mathrm{Inv}(\mathbf{a_3}\pi_\mathbf{0})$.

Proof. Obviously, $\{0,1\} = \mathrm{Strike}(\rho_W, 2)$. Hence, by Lemma 3.1 we have $B_0 \subseteq [\{\rho_W, \rho_{+1}\}]$. Let $\sigma = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. Then

$$\sigma(x,y) = \sigma_3^{=}(x,y) \wedge (x \in \{0,1\}),$$

$$\rho_{\le}(x,y) = \rho_W(y, x+1) \wedge (x \in \{0,1\}),$$

$$\rho_{\vee,2}(x,y) = \rho_W(y, x+2) \wedge (x \in \{0,1\}),$$

$$\rho_N(x,y) = \rho_W(x,y) \wedge \rho_W(x, y+1).$$

□

Lemma 3.6. $B_0 \cup B_1 \subseteq [\{\rho_{=,01}, \rho_{+1}\}]$.

Remark. Note that $[\{\rho_{=,01}, \rho_{+1}\}] = \mathrm{Inv}(\mathbf{aQ})$.

Proof. Obviously, $\rho_W(x,y) = \exists z\ \rho_{=,01}(x,y,z)$, hence by Lemma 3.5 we have $B_0 \cup B_1 \subseteq [\{\rho_{=,01}, \rho_{+1}\}]$.

□

Lemma 3.7. $B_0 \subseteq [\{\rho_{=,012}, \rho_{+1}\}]$.

Remark. Note that $[\{\rho_{=,012}, \rho_{+1}\}] = \mathrm{Inv}(\mathbf{aW})$.

Proof. Obviously, we have $\{0,1\} = \mathrm{Strike}(\mathrm{Strike}(\rho_{=,012}, 2), 2)$. Therefore, by Lemma 3.1 we get $B_0 \subseteq [\{\rho_{=,012}, \rho_{+1}\}]$.

□

Lemma 3.8. *Suppose* $\rho \in \mathrm{Inv}(right)$, *then* $\rho \in [\{\rho_{=,012}, \rho_{+1}\}]$.

Remark. This lemma states that $\mathrm{Inv}(right) \subseteq [\{\rho_{=,012}, \rho_{+1}\}] = \mathrm{Inv}(\mathbf{aW})$, that is $[\{right\}] \supseteq \mathbf{aW}$.

Proof. The proof is by induction on the arity of ρ. If $ar(\rho) = 0$ then the proof is trivial.

By Lemma 2.6, there exists a set of predicates $W \subseteq [\{\rho\} \cup \{\rho_{=,012}, \rho_{+1}\}]$ such that $\rho \in \mathrm{And}(S)$ and every predicate in W is maximal with respect to $\{\rho_{=,012}, \rho_{+1}\}$. Since $right$ preserves the predicates $\rho_{=,012}$ and ρ_{+1}, we have $W \subseteq \mathrm{Inv}(right)$. Hence, it is sufficient to prove this lemma only for predicates that are maximal with respect to $\{\rho_{=,012}, \rho_{+1}\}$. Let α be a key word for ρ. By Lemma 3.4 it can be assumed that $\rho \in B_0 \cup Main$. If $\rho \in B_0$, then the proof follows from Lemma 3.7. Suppose $\rho \in Main$, $\mathrm{VarValues}(\rho, i) = \{0, 1\}$ for $i \leq m$, $\mathrm{VarValues}(\rho, i) = E_3$ for $i > m$, $n = \mathrm{ar}(\rho) - m$.

By the definition of $Main$, there exists $i \leq m$ such that $\alpha(i) = 0$. Without loss of generality it can be assumed that $i = 1$. Let

$$\rho_0(x_2, x_3, \ldots, x_m, y_1, \ldots, y_n) = \rho(0, x_2, x_3, \ldots, x_m, y_1, \ldots, y_n)$$

By Lemma 2.3 we have $\rho_0 \in [\{\rho, \{0\}\}]$. Hence $\rho_0 \in \mathrm{Inv}(right)$, and by the inductive assumption we obtain $\rho_0 \in [\{\rho_{=,012}, \rho_{+1}\}]$.

Let us define a predicate ρ_α. Put $\rho_\alpha(x_1, \ldots, x_{m+n}) = 1$ iff there exist $f_2, f_3, \ldots, f_{m+n} \in E_3$ such that for every $i \in \{2, 3, \ldots, m+n\}$ we have

$$\rho_{=,012}(x_1, f_i, x_i) = 1,$$

$$\rho_0(f_2, f_3, \ldots, f_{m+n}) = 1.$$

Obviously, $\rho_\alpha \in [\{\rho_0, \rho_{=,012}\}] \subseteq [\{\rho_{=,012}, \rho_{+1}\}]$, $\rho_\alpha(\alpha) = 0$, and $\rho_\alpha \geq \rho$. Since α is a key word for ρ and ρ is maximal with respect to $\{\rho_{=,012}, \rho_{+1}\}$, we have $\rho_\alpha = \rho$. This completes the proof.

□

Lemma 3.9. *Suppose* $\rho \in \mathrm{Inv}(right)$, $\rho_{=,012} \notin [\{\rho, \rho_{=,01}, \rho_{+1}\}]$. *Then* $\rho \in [\{\rho_{=,01}, \rho_{+1}\}]$.

Remark. In other words, if C is a clone, $\mathbf{aW} \subseteq C \subseteq \mathbf{aQ}$, and $C \neq \mathbf{aW}$, then $C = \mathbf{aQ}$.

Proof. The proof is by induction on the arity of ρ. If $ar(\rho) = 0$ then the proof is trivial.

By Lemma 2.6, it is sufficient to prove this lemma only for predicates that are maximal with respect to $\{\rho_{=,01}, \rho_{+1}\}$. Let α be a key word for ρ. By Lemma 3.4 $\rho \in \mathrm{Shift}(B_0 \cup Main)$. If $\rho \in \mathrm{Shift}(B_0)$, then the proof follows from Lemma 3.6. Suppose $\rho \in \mathrm{Shift}(Main)$, then there exists i such that

$\mathrm{VarValues}(\rho, i) = \{b, b+1\}$ and $\alpha(i) = b$. Without loss of generality it can be assumed that $\mathrm{VarValues}(\rho, 1) = \{0, 1\}$ and $\alpha(1) = 0$.

Let $r = \mathrm{ar}(\rho)$, $\rho_0(x_2, x_3, \ldots, x_r) = \rho(0, x_2, x_3, \ldots, x_r)$. By Lemma 2.3 and Lemma 3.6 it follows that $\rho_0 \in [\{\rho, \{0\}\}] \subseteq [\{\rho, \rho_{=,01}, \rho_{+1}\}]$. Hence, $\rho_0 \in \mathrm{Inv}(right)$, $\rho_{=,012} \notin [\{\rho_0, \rho_{=,01}, \rho_{+1}\}]$ and by the inductive assumption we obtain $\rho_0 \in [\{\rho_{=,01}, \rho_{+1}\}]$.

Let $\alpha' = [_{r-1}(\alpha)$. It can be easily checked that $\rho_0(\alpha') = 0$ and α' is an essential word for ρ_0. Hence, ρ_0 is an essential predicate.

Then by Lemma 3.4 it can be assumed that $\rho_0 \in B_0 \cup Main$.

Suppose $\mathrm{ar}(\rho_0) = 1$, then ρ belongs to the set

$$\mathrm{Shift}\left(\left\{\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\end{pmatrix}, \rho_N, \rho_W\right\}\right).$$

Hence, by Lemma 3.6 we have $\rho \in [\{\rho_{=,01}, \rho_{+1}\}]$.

Suppose $\mathrm{ar}(\rho_0) = 2$ and $\rho_0 \in B_0$, then $\rho \in \mathrm{Shift}(\rho_{=,012})$ and $\rho_{=,012} \in [\{\rho, \rho_{+1}\}]$. This contradict the assumption about ρ.

Suppose that $\rho_0 \in Main$. Let $\mathrm{VarValues}(\rho_0, i) = \{0, 1\}$ for $i \leq m$, $\mathrm{VarValues}(\rho_0, i) = E_3$ for $i > m$, $n = \mathrm{ar}(\rho_0) - m$.

Let us define a predicate ρ_α. Put $\rho_\alpha(x_1, x_2, \ldots, x_{m+n+1}) = 1$ iff there exist $f_1, f_2, \ldots, f_m \in E_3$ such that for every $i \in \{1, 2, \ldots, m\}$ we have

$$\rho_{=,01}(x_1, f_i, x_{i+1}) = 1,$$

$$\rho_0(f_1, \ldots, f_m, x_{m+2}, \ldots, x_{m+n+1}) = 1.$$

Obviously, $\rho_\alpha \in [\{\rho_0, \rho_{=,01}\}] \subseteq [\{\rho_{=,01}, \rho_{+1}\}]$, $\rho_\alpha(\alpha) = 0$, and $\rho_\alpha \geq \rho$. Since α is a key word for ρ and ρ is maximal with respect to $\{\rho_{=,01}, \rho_{+1}\}$, we have $\rho_\alpha = \rho$. This completes the proof.

□

3.3 Clones $\mathbf{aP}, \mathbf{aPN}, \mathbf{aP_n}$.

By $\rho_{key,1}$, $\rho_{key,2}$, and $\rho_{key,3}$ we denote the predicates defined by the following conditions:

$$\rho_{key,1}(x_1, x_2, x_3) = 1 \Leftrightarrow (x_1, x_2 \in \{0, 1\}) \wedge ((x_1 = 1) \vee ((x_2 = 0) \wedge x_3 \in \{0, 1\}) \vee (x_3 = 1)),$$

$$\rho_{key,2}(x_1, x_2, x_3) = 1 \Leftrightarrow (x_1 = 1) \vee ((x_1 = 0) \wedge (x_2 = 1)) \vee ((x_1 = 0) \wedge (x_2 = 0) \wedge (x_3 = 1)),$$

$$\rho_{key,3}(x_1,x_2,x_3)=1 \Leftrightarrow (x_1=1)\vee((x_1=0)\wedge(x_2=1))\vee \\ ((x_1=0)\wedge(x_2=0)\wedge(x_3\in\{1,2\})).$$

Lemma 3.10. $\{\rho_{\leq},\rho_{\vee,2}\}\subseteq[\{\rho_N,\rho_{+1}\}]$.

Remark. In other words, $[\{\rho_N,\rho_{+1}\}]=[\{\rho_N,\rho_{\vee,2},\rho_{+1}\}]=\mathrm{Inv}(\mathbf{a_2N})$ and $\mathbf{a_2N}\subseteq\mathbf{M}$.

Proof. Obviously $\{0,1\}=\mathrm{Strike}(\rho_N,2)$. Then we have

$$\rho_{\leq}(x,y)=\rho_N(y,x)\wedge(x\in\{0,1\}),$$

$$\rho_{\vee,2}(x,y)=\rho_N(x,y+2)\wedge(y\in\{0,1\}).$$

□

Lemma 3.11. $\rho_{\vee,3}=\pi_{\varnothing,\varnothing,\varnothing}\in[\{\rho_W,\rho_{+1}\}]$.

Remark. In other words, this lemma states that $\mathrm{Pol}(\{\rho_W,\rho_{+1}\})=\mathbf{a_3}\pi_\mathbf{0}$

Proof. $\rho_{\vee,3}(x_1,x_2,x_3)=\exists y\ \rho_W(x_1,y)\wedge\rho_W(x_2,y+1)\wedge\rho_W(x_3,y+2)$. □

Lemma 3.12. *Suppose* $\rho\in\Pi_n^m$, $m+n\geq 3$, $n\geq 1$. *Then* $\rho_W\in[\{\rho\}]$.

Proof. Suppose $\rho=\pi_{A_1,\ldots,A_m}$ and $1\in A_i$. It can be easily checked that ρ_W is obtained from ρ by striking all rows except the i-th and $(m+1)$-th.

□

Lemma 3.13. $\pi_{\{1,2,\ldots,n\}}\in[\{\pi_{\{1,2,\ldots,n,n+1\}}\}]$.

Remark. This lemma implies that $\mathbf{aP_{n+1}}\subseteq\mathbf{aP_n}$.

Proof. $\pi_{\{1,2,\ldots,n\}}(x,y_1,y_2,\ldots,y_n)=\pi_{\{1,2,\ldots,n,n+1\}}(x,y_1,y_1,y_2,\ldots,y_n)$ □

Lemma 3.14. $\rho_{=,01}\in[\{\rho_{key,1}\}\cup B_0]$.

Proof. Obviously, $\rho_W(x,z)=\exists y\ \rho_{key,1}(x,y,z)$,

$$\rho_{=,01}(x_1,x_2,x_3)=\exists y\ \rho_{key,1}(x_1,y,x_2)\wedge\rho_W(y,x_2+1)\wedge \\ \rho_{key,1}(x_1,y,x_3)\wedge\rho_W(y,x_3+1).$$

□

Lemma 3.15. $\rho_{=,01} \in [\{\rho_{key,2}\} \cup B_0]$.

Proof. $\rho_{=,01}(x, y, z) = \rho_{key,2}(x, y, z + 1) \wedge \rho_{key,2}(x, z, y + 1)$.

□

Lemma 3.16. $\rho_{=,01} \in [\{\rho_{key,3}\} \cup B_0]$.

Proof. $\rho_{=,01}(x, y, z) = \rho_{key,3}(x, y, z + 2) \wedge \rho_{key,3}(x, z, y + 2)$.

□

Let $\kappa_0 = \rho_{\leq}$, $\kappa_1 = \rho_N$, $\kappa_2 = \rho_W$, $\kappa_r = \pi_{\{1,2,\ldots,r-1\}}$ for $r \geq 3$.

By Lemma 3.10, Lemma 3.5, Lemma 3.12, and Lemma 3.13 it follows that $\kappa_r \in [\{\kappa_{r+1}, \rho_{+1}\}]$ for every $r \geq 0$.

Lemma 3.17. *Suppose $\rho \in \mathrm{Inv}(right)$, $\rho_{=,01} \notin [\{\rho\} \cup B_0 \cup B_2]$. Then there exists $r \leq \mathrm{ar}(\rho)$ such that ρ is equivalent to κ_r with respect to $B_0 \cup B_2$.*

Remark. The lemma essentially shows that if C is a clone and $\mathbf{aQ} \subset C \subseteq \mathbf{aP}$ then $C \in \{\mathbf{aP}_\infty, \mathbf{aP}, \mathbf{aPN}, \mathbf{aP_1}, \mathbf{aP_2}, \mathbf{aP_3}, \ldots,\}$.

Proof. The proof is by induction on the arity of ρ. If $\mathrm{ar}(\rho) = 0$ then ρ is equivalent to κ_0 with respect to $B_0 \cup B_2$. By Lemma 2.6, ρ is presented as a conjunction of predicates $\delta_1, \ldots, \delta_s \in [\{\rho\} \cup B_0 \cup B_2]$ such that δ_i is maximal with respect to $B_0 \cup B_2$ for every $i \in \{1, 2, \ldots, s\}$. Suppose we prove that for every $i \in \{1, 2, \ldots, s\}$ there exists r_i such that δ_i is equivalent to κ_{r_i} with respect to $B_0 \cup B_2$. Therefore, it can be easily checked that ρ is equivalent to κ_r with respect to $B_0 \cup B_2$, where $r = \max_i(r_i)$.

Hence, it is sufficient to prove this lemma only for predicates that are maximal with respect to $B_0 \cup B_2$. Let α be a key word for ρ. By Lemma 3.4 it can be assumed that $\rho \in B_0 \cup Main$. If $\rho \in B_0$, then ρ is equivalent to $\kappa_0 = \rho_{\leq} \in B_2$ with respect to $B_0 \cup B_2$. Suppose $\rho \in Main$, $\mathrm{VarValues}(\rho, i) = \{0, 1\}$ for $i \leq m$, $\mathrm{VarValues}(\rho, i) = E_3$ for $i > m$, $n = \mathrm{ar}(\rho) - m$. Since α is essential for ρ, for every $j \in \{1, 2, \ldots, m + n\}$ there exists $\beta_j \in \rho$ such that $\alpha(p) = \beta_j(p)$ for every $p \neq j$.

If $n = 0$, then $\rho \in B_2$ and ρ is equivalent to κ_0 with respect to $B_0 \cup B_2$.

Suppose $n \geq 1$. If $m = n = 1$, then $\rho \in \mathrm{Shift}(\{\kappa_1, \kappa_2\})$.

Suppose $m + n \geq 3$. Assume that there exist $i_1, i_2 \leq m$, $i_1 \neq i_2$ such that $\alpha(i_1) = \alpha(i_2) = 1$. Since $\rho \in \mathrm{Inv}(right)$, we see that $right(\beta_{i_1}, \beta_{i_2}) = \alpha \in \rho$. This contradiction proves that $]_m(\alpha)$ contains at most one 1.

Let us consider two cases. First case, there exists $j \in \{1, 2, \ldots, m\}$ such that $\alpha(j) = 1$. Without loss of generality it can be assumed that $j = 1$. Using Lemma 2.2, it can be assumed that $\alpha(p) = 0$ for every $p \geq m + 1$. Then we have $\alpha = 10^{m+n-1}$. Suppose $\beta_i(i) = 2$ for some $i \geq m+1$, then $right(\beta_i, \beta_1) = \alpha \in \rho$. This contradiction proves that $\beta_j \in \{0,1\}^{m+n}$ for every $j \in \{1, 2, \ldots, m+n\}$.

Let us prove that $\rho_{key,1} \in [\{\rho\} \cup B_0 \cup B_2]$. Assume that $d0^{m+n-2}2 \in \rho$ for some $d \in \{0,1\}$. Since $10^{m+n-2}1 = \beta_{m+n}$, $0^{m+n} = \beta_1$, we have

$$right(right(d0^{m+n-2}2, 10^{m+n-2}1), 0^{m+n}) = 10^{m+n-1} = \alpha \in \rho.$$

This contradiction proves that $d0^{m+n-2}2 \notin \rho$ for every $d \in \{0,1\}$. Then

$$\rho_{key,1}(x_1, x_2, x_3) = \exists f \ \rho_{\leq}(x_2, f) \wedge x_1 \in \{0,1\} \wedge \rho(f, x_1, \ldots, x_1, x_3).$$

Since $\{0,1\}, \rho_{\leq} \in B_0 \cup B_2$, we see that $\rho_{key,1} \in [\{\rho\} \cup B_0 \cup B_2]$. Hence, by Lemma 3.14 we have $\rho_{=,01} \in [\{\rho\} \cup B_0 \cup B_2]$. This contradicts the condition of the lemma.

Second case, $\alpha(j) = 0$ for every $j \leq m$. Assume that $m \geq 2$. Let

$$\rho'(x, y_1, \ldots, y_n) = \rho(x, x, \ldots, x, y_1, \ldots, y_m).$$

By the inductive assumption, ρ' is equivalent to κ_p with respect to $B_0 \cup B_2$ for some $p \geq 0$. Let us define a predicate ρ_α

$$\rho_\alpha(x_1, \ldots, x_m, y_1, \ldots, y_n) = \exists f \ \rho_{\rightarrow,m}(x_1, \ldots, x_m, f) \wedge \rho'(f, y_1, \ldots, y_n).$$

It can be easily checked that $\rho'(x, y_1, \ldots, y_n) = \rho_\alpha(x, x, \ldots, x, y_1, \ldots, y_m)$. Hence, ρ_α and ρ' are equivalent with respect to $B_0 \cup B_2$. Obviously, $\rho_\alpha \geq \rho$ and $\rho_\alpha(\alpha) = 0$. Since ρ is maximal with respect to $B_0 \cup B_2$ we obtain $\rho = \rho_\alpha$. Then ρ is equivalent to κ_p with respect to $B_0 \cup B_2$.

Assume that $m = 1, n \geq 2$. By Lemma 2.2, it can be assumed that $\alpha, \beta_1, \ldots, \beta_{n+1} \in \{0,1\}^{n+1}$. Assume that $\alpha(i_1) = \alpha(i_2) = 1$ for some $i_1, i_2, i_1 \neq i_2$. Since $\rho \in \mathrm{Inv}(right)$ we have $right(\beta_{i_1}, \beta_{i_2}) = \alpha \in \rho$. This contradiction proves that either there exists a unique i such that $\alpha(i) = 1$, or $\alpha = 0^{n+1}$. Let us consider two subcases. First subcase, $\alpha \neq 0^{n+1}$. Without loss of generality it can be assumed that $\alpha = 0^n1$. Hence,

$$right(right(\ldots(right(\beta_2, \beta_3), \beta_4), \ldots), \beta_n) = 01^n \in \rho.$$

Obviously, $\beta_{n+1} = 0^{n+1} \in \rho$. Assume that $02^{n-1}1 \in \rho$, then

$$right(02^{n-1}1, 0^{n+1}) = 0^n1 = \alpha \in \rho.$$

This contradiction proves that $02^{n-1}1 \notin \rho$.

Assume that $02^{n-1}0 \in \rho$, then

$$right(02^{n-1}0, 01^n) = 02^{n-1}1 \in \rho.$$

This contradiction proves that $02^{n-1}0 \notin \rho$. Let

$$\delta(x_1, x_2, x_3) = \rho(x_1, x_2, \ldots, x_2, x_3) \wedge \rho(x_1, x_3, \ldots, x_3, x_2).$$

It can be easily checked that if $\rho(0, 2, \ldots, 2) = 1$, then $\delta = \rho_{=,012}$; and if $\rho(0, 2, \ldots, 2) = 0$, then $\delta = \rho_{=,01}$. Then it follows from Lemma 3.8 that $\rho_{=,01} \in [\{\rho\} \cup B_0 \cup B_2]$. This contradicts the condition of the lemma.

Second subcase, $\alpha = 0^{n+1}$. Hence $\beta_i = 0^{i-1}10^{n-i+1}$ for every i. Since $\rho \in \mathrm{Inv}(right)$ we see that for every $a_1, \ldots, a_{n+1} \in \{0, 1\}$

$$\rho(a_1, a_2, \ldots, a_{n+1}) = 0 \Leftrightarrow a_1 = a_2 = \ldots = a_{n+1} = 0.$$

Let $\rho_0(y_1, \ldots, y_n) = \rho(0, y_1, \ldots, y_n)$. Obviously, 0^n is an essential word for ρ_0 and ρ_0 is essential. By Lemma 3.4 we have $\rho_0 \in \mathrm{Shift}(B_0 \cup Main)$.

If $\rho_0 \in \mathrm{Shift}(B_0)$, then $\rho \in \mathrm{Shift}(\rho_{=,012})$. By Lemma 3.8 we have $\rho_{=,01} \in [\{\rho\} \cup B_0 \cup B_2]$. This contradicts the condition of the lemma.

Suppose $\rho_0 \in \mathrm{Shift}(Main)$. Let $W_1 = \{i \mid \mathrm{VarValues}(\rho_0, i) = E_3\}$, $W_2 = \{1, 2, \ldots, n\} \setminus W_1$. Assume that $W_1 \neq \varnothing$. Substituting in ρ variable y for the $(i+1)$-th variable for every $i \in W_2$, and substituting variable z for the $(i+1)$-th variable for every $i \in W_1$ we obtain a predicate $\rho'(x_1, y, z)$ such that:

$$\forall c, d \in E_3 \quad \rho'(1, c, d) = \rho'(0, 1, c) = \rho'(0, 0, 1) = 1,$$

$$\forall c, d \in E_3 \quad \rho'(2, c, d) = \rho'(0, 2, c) = \rho'(0, 0, 0) = 0.$$

If $\rho'(0, 0, 2) = 0$, then $\rho' = \rho_{key,2}$; if $\rho'(0, 0, 2) = 1$, then $\rho' = \rho_{key,3}$. By Lemma 3.15 and Lemma 3.16 it follows that $\rho_{=,01} \in [\{\rho\} \cup B_0 \cup B_2]$. This contradicts the condition of the lemma.

Suppose $W_1 = \varnothing$. Therefore, $\rho_0 = \rho_{\vee,n}$. It can be easily checked that $\rho = \pi_{\{1,\ldots,n\}} = \kappa_{n+1}$. This completes the proof.

□

3.4 Final constructions for Φ and Υ.

Lemma 3.18. *Suppose $\rho \in B_2$, α is an essential word for ρ. Then α contains at most one 1.*

Proof. Suppose $n = \mathrm{ar}(\rho)$. By the definition of an essential tuple for every $i \in \{1, 2, \ldots, n\}$ there exists $\beta_i \in \rho$ such that $\alpha(j) = \beta_i(j)$ for every $j \neq i$.

Assume the converse. Let $\alpha(i) = \alpha(j) = 1$ for some $i \neq j$. Hence $right(\beta_i, \beta_j) = \alpha \in \rho$. This contradiction completes the proof.

□

Lemma 3.19. $B_2 \subseteq [\{\rho_{\to,2}, \{0\}, \{1\}\}]$.

Proof. We consider all predicates from B_2 as predicates from R_2. It is sufficient to show that $\mathrm{Pol}(\{\rho_{\to,2}, \{0\}, \{1\}\}) \subseteq \mathrm{Pol}(B_2)$. Obviously $x \vee y$ preserves every predicate $\sigma \in B_2$. By the description of Post's lattice we have $\mathrm{Pol}(\{\rho_{\to,2}, \{0\}, \{1\}\}) = \mathrm{Pol}(B_2) = [\{x \vee y\}]$. This completes the proof.

□

Lemma 3.20. *Suppose $\rho \in B_2$ is essential,* $\mathrm{ar}(\rho) \geq 2$*,* $\rho_{\to,2} \notin [\{\rho\}]$*, then*

$$\rho \in \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \rho_{\vee,2}, \rho_{\vee,3}, \dots \right\}$$

Proof. If $\mathrm{ar}(\rho) = 2$, then we can easily show that

$$\rho \in \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \rho_{\vee,2} \right\}.$$

Suppose $\mathrm{ar}(\rho) \geq 3$. Let α be an essential word for $\rho \in R_3^n$. For every $i \in \{1, 2, \dots, n\}$ there exists $\beta_i \in \rho$ such that $\alpha(j) = \beta_i(j)$ for every $j \neq i$.

By Lemma 3.18 α contains at most one 1. We consider two cases. First case, $\alpha(j) = 1$ for some $j \in \{1, 2, \dots, n\}$. Without loss of generality it can be assumed that $j = 1$. Then we have $\beta_1 = 0^n$, $\beta_i = 10^{i-2}10^{n-i}$ for $i \geq 2$. Let us prove that $\rho_{\leq} \in [\{\rho\}]$. Since $\rho \in \mathrm{Inv}(right)$, we have $1\gamma \in \rho$ for every $\gamma \in \{0, 1\}^{n-1} \setminus \{0^{n-1}\}$.

We consider two subcases. First subcase, there exists $i \in \{2, 3 \dots, n\}$ such that $0^{i-1}10^{n-i} \notin \rho$. Without loss of generality it can be assumed that $i = n$. Then $\rho_{\leq}(x, y) = \rho(y, y, \dots, y, x)$

Second subcase, for every $i \in \{2, 3 \dots, n\}$ we have $0^{i-1}10^{n-i} \in \rho$. Since $\rho \in \mathrm{Inv}(right)$, we get $01^{n-1} \in \rho$. Then $\rho_{\leq}(x, y) = \rho(x, y, y, \dots, y)$

So, we proved that $\rho_{\leq} \in [\{\rho\}]$. It can be easily checked that

$$\rho_{\to,2}(x, y, z) = \exists x' \, \exists y' \, \rho(z, x', y', y', \dots, y') \wedge \rho_{\leq}(x', x) \wedge \rho_{\leq}(y', y).$$

But this contradicts the condition of the lemma.

Second case, $\alpha = 0^n$. It can easily be checked that $\beta_i = 0^{i-1}10^{n-i}$ for every $i \in \{1, 2, \dots, n\}$. Since $\rho \in \mathrm{Inv}(right)$, we obtain $\rho = \rho_{\vee,n}$.

□

Lemma 3.21. *Suppose* $\rho \in \mathrm{Inv}(right) \cap \widetilde{R}_3$, $\rho_{\rightarrow,2} \notin [\{\rho\} \cup B_0]$, *then* $\rho \in \mathrm{Shift}(B_0 \cup B_1 \cup \Pi)$.

Proof. Since ρ is essential, there exists an essential word α for ρ. By the definition of an essential tuple for every $i \in \{1, 2, \dots, \mathrm{ar}(\rho)\}$ there exists $\beta_i \in \rho$ such that $\alpha(j) = \beta_i(j)$ for every $j \neq i$.

By Lemma 3.4 we can assume that $\rho \in B_0 \cup Main$. If $\rho \in B_0$ then there is nothing to prove. Suppose $\rho \in Main$, which means that $\mathrm{VarValues}(\rho, i) = \{0, 1\}$ for $i \leq m$, $\mathrm{VarValues}(\rho, i) = E_3$ for $i > m$, $n = \mathrm{ar}(\rho) - m$. By Lemma 2.2, we can assume that $\alpha, \beta_1, \dots, \beta_{m+n} \in \{0, 1\}^{m+n}$.

If $n = 0$, then by Lemma 3.20 we have $\rho \in B_0 \cup B_1 \cup \Pi_0$, which proves the lemma in this case.

Suppose $n \geq 1$. If $m = n = 1$, then $\rho \in \mathrm{Shift}(\{\rho_N, \rho_W\}) \subseteq \mathrm{Shift}(B_1)$, which proves the lemma in this case.

Suppose $m + n \geq 3$. Let

$$\rho_0(x_1, \dots, x_m, y_1, \dots, y_n) = 1 \Leftrightarrow \\ (\rho(x_1, \dots, x_m, y_1, \dots, y_n) = 1) \wedge (\forall i\ y_i \in \{0, 1\}).$$

In other words $\rho_0 = \rho \cap \{0, 1\}^{m+n}$. Obviously, $\rho_0 \in [\{\rho\} \cup B_0]$. Clearly, α is an essential word for ρ_0, hence, ρ_0 is essential and by Lemma 3.20 we have $\rho_0 = \rho_{\vee, m+n}$. Therefore, $\alpha = 0^{m+n}$.

Let

$$\rho_1(x_1, \dots, x_m, y_1, \dots, y_n) = 1 \Leftrightarrow \\ (\rho(x_1, \dots, x_m, y_1 + 1, \dots, y_n + 1) = 1) \wedge (\forall i\ y_i \in \{0, 1\}).$$

Obviously $\rho_1 \in [\{\rho\} \cup B_0]$. It follows from Lemma 2.5 and Lemma 3.20 that

$$\rho_1 \in \mathrm{And}(\{\sigma, \rho_{\leq}, \{0\}, \{1\}, \rho_{\vee,2}, \rho_{\vee,3}, \dots\}),$$

where $\sigma = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Since $\rho_0 = \rho_{\vee, m+n}$ and $n \geq 1$ we get $\rho_1(a_1, \dots, a_m, 0, \dots, 0) = 1$ for every $a_1, \dots, a_m \in \{0, 1\}$. Since $\rho \in Main$ for every $b_1, \dots, b_n \in \{0, 1\}$ we have $\rho_1(1, \dots, 1, b_1, \dots, b_n) = 1$.

Therefore, $\rho_1 \in \mathrm{And}(\{\rho_{\leq}\})$ and there exist $i_1, \dots, i_s \in \{1, 2, \dots, n\}$, $j_1, \dots, j_s \in \{1, 2, \dots, m\}$ such that

$$\rho_1(x_1, \dots, x_m, y_1, \dots, y_n) = \rho_{\leq}(y_{i_1}, x_{j_1}) \wedge \dots \wedge \rho_{\leq}(y_{i_s}, x_{j_s}).$$

By A_j we denote the set of all i such that $\rho_{\leq}(y_i, x_j)$ is used in this formula.

Assume that $A_1 \cup \ldots \cup A_m \neq \{1, 2 \ldots, n\}$. Then there exists some

$$c \in \{1, 2 \ldots, n\} \setminus (A_1 \cup \ldots \cup A_m).$$

Let

$$\rho_2(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1 \Leftrightarrow (\forall i \; y_i \in \{0, 1\}) \wedge \\ (\rho(x_1, \ldots, x_m, y_1, \ldots, y_{c-1}, y_c + 1, y_{c+1}, \ldots, y_n) = 1).$$

Obviously $\rho_2 \in [\{\rho\} \cup B_0]$. Analogously as for ρ_1 we show that for every $a_1, \ldots, a_m, b_1, \ldots, b_n \in \{0, 1\}$ we have

$$\rho_2(a_1, \ldots, a_m, 0, \ldots, 0) = \rho_2(1, \ldots, 1, b_1, \ldots, b_n) = 1.$$

Since $\rho_1(\underbrace{0, \ldots, 0}_{m+c-1}, 1, \underbrace{0, \ldots, 0}_{n-c}) = 1$, we get $\rho_2(\underbrace{0, \ldots, 0}_{m}, \underbrace{1, \ldots, 1}_{n}) = 1$. By Lemma 2.5 and Lemma 3.20 we have

$$\rho_2 \in \mathrm{And}(\{\sigma, \rho_{\leq}, \{0\}, \{1\}, \rho_{\vee,2}, \rho_{\vee,3}, \ldots\}).$$

Hence, we get $\rho_2 = \{0, 1\}^{m+n}$.

Let

$$\rho_3(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1 \Leftrightarrow (\forall i \quad y_i \in \{0, 1\}) \wedge \\ (\rho(x_1, \ldots, x_m, y_1, \ldots, y_{c-1}, y_c + 2, y_{c+1}, \ldots, y_n) = 1).$$

Obviously, we have $\rho_3 \in [\{\rho\} \cup B_0]$. Since $\rho_2 = \{0, 1\}^{m+n}$, for every $a_1, \ldots, a_m, b_1, \ldots, b_n \in \{0, 1\}$ we have

$$\rho_3(a_1, \ldots, a_m, 0, \ldots, 0) = \rho_3(1, \ldots, 1, b_1, \ldots, b_n) = 1.$$

Since $\rho_0 = \rho_{\vee,m+n}$ and $m + n \geq 3$, for every $j \in \{1, \ldots, m\}$ we have

$$\rho_3(\underbrace{1, 1, \ldots, 1}_{j-1}, 0, \underbrace{1, 1, \ldots, 1}_{m+n-j}) = 1.$$

By Lemma 2.5 and Lemma 3.20 it follows that

$$\rho_3 \in \mathrm{And}(\{\sigma, \rho_{\leq}, \{0\}, \{1\}, \rho_{\vee,2}, \rho_{\vee,3}, \ldots\}).$$

Hence, we get $\rho_3 = \{0, 1\}^{m+n}$, and $\rho(0, \ldots, 0) = 1$. This contradiction proves that $A_1 \cup \ldots \cup A_m = \{1, 2 \ldots, n\}$.

Let us prove that $\rho = \pi_{A_1,\ldots,A_m}$. Obviously these predicates are equal on the tuples from $\{0,1\}^{m+n}$. Let $\gamma \in \{0,1\}^m \times E_3^n$, $\gamma(j) = 2$ for some j. Without loss of generality it can be assumed that there exists $n' < n$ such that

$$\gamma(m+1), \gamma(m+2), \ldots, \gamma(m+n') \in \{0,1\},$$

$$\gamma(m+n'+1) = \gamma(m+n'+2) = \ldots = \gamma(m+n) = 2.$$

Put $\gamma' =]_m(\gamma)1^{n'}2^{n-n'}$. Let

$$\rho_4(x_1,\ldots,x_m,y_1,\ldots,y_n) = 1 \Leftrightarrow (\forall i \quad y_i \in \{0,1\}) \wedge \\ (\rho(x_1,\ldots,x_m,y_1,\ldots,y_{n'},y_{n'+1}+1,\ldots,y_n+1) = 1).$$

Obviously $\rho_4 \in [\{\rho\} \cup B_0]$. In the same way as for the predicates ρ_1 and ρ_2 we can show that for every $a_1,\ldots,a_{m+n'},b_1,\ldots,b_n \in \{0,1\}$ we have

$$\rho_4(a_1,\ldots,a_{m+n'},0,\ldots,0) = \rho_4(1,\ldots,1,b_1,\ldots,b_n) = 1.$$

By Lemma 2.5 and Lemma 3.20 we have

$$\rho_4 \in \mathrm{And}(\{\sigma, \rho_{\leq}, \{0\}, \{1\}, \rho_{\vee,2}, \rho_{\vee,3}, \ldots\}).$$

Hence, $\rho_4 \in \mathrm{And}(\{\rho_{\leq}\})$ and there exist $d_1,\ldots,d_{s'} \in \{n'+1, n'+2, \ldots, n\}$, $e_1,\ldots,e_{s'} \in \{1,2,\ldots,m\}$ such that

$$\rho_4(x_1,\ldots,x_m,y_1,\ldots,y_n) = \rho_{\leq}(y_{d_1},x_{e_1}) \wedge \ldots \wedge \rho_{\leq}(y_{d_{s'}},x_{e_{s'}}).$$

Using the formula above, we get $\rho(\gamma) = \rho(\gamma')$. Obviously,

$$\pi_{A_1,\ldots,A_m}(\gamma) = \pi_{A_1,\ldots,A_m}(\gamma').$$

By the definition of $A_1,\ldots,A_m$ we have $\rho(\gamma') = \pi_{A_1,\ldots,A_m}(\gamma')$. Hence, $\pi_{A_1,\ldots,A_m}(\gamma) = \rho(\gamma)$ for every $\gamma \in \{0,1\}^m \times E_3^n$. This completes the proof. □

Lemma 3.22. *Suppose* $\rho_1, \rho_2 \in \Pi$, $\rho_2 \lesssim^1 \rho_1$, *then* $\rho_2 \in [\{\rho_1\} \cup B_0 \cup B_1]$.

Proof. Suppose $\rho_1 = \pi_{A_1,\ldots,A_m} \in \Pi_n^m$, $\rho_2 = \pi_{A'_1,\ldots,A'_{m'}} \in \Pi_{n'}^{m'}$, where $m' \geq m$, $n' \leq n$, $m'+n' = m+n$, $A'_i = A_i \cap \{1,2,\ldots,n'\}$ for $i \in \{1,2,\ldots,m\}$, $A'_i = \varnothing$ for $i \in \{m+1,m+2,\ldots,m'\}$. It can be easily checked that

$$\rho_2(x_1,\ldots,x_{m'},y_1,\ldots,y_{n'}) = (\forall i \geq m+1: \quad x_i \in \{0,1\}) \wedge \\ \rho_1(x_1,\ldots,x_m,y_1,\ldots,y_{n'},x_{m+1},x_{m+2},\ldots,x_{m'}).$$

□

Lemma 3.23. *Suppose* $\rho_1, \rho_2 \in \Pi$, $\rho_2 \lesssim^2 \rho_1$, *then* $\rho_2 \in [\{\rho_1\} \cup B_0 \cup B_1]$.

Proof. Suppose that $\rho_1 = \pi_{A_1,\dots,A_m} \in \Pi_n^m$, $\rho_2 = \pi_{A'_1,\dots,A'_{m'}} \in \Pi_{n'}^{m'}$, $m' \leq m$, $n' = n$, the set $\{1, 2, \dots, m\}$ is divided into non-overlapping sets $K_1, \dots, K_{m'}$ such that $A'_i = \bigcup_{j \in K_i} A_j$. It is easy to show that ρ_2 can be obtained from $\rho_1(x_1, \dots, x_m, y_1, \dots, y_n)$ by identification of variables from the set $\{x_i \mid i \in K_j\}$ for every j, and permutation of variables.

□

Lemma 3.24. *Suppose* $\rho_1, \rho_2 \in \Pi$, $\rho_2 \lesssim^3 \rho_1$, *then* $\rho_2 \in [\{\rho_1\} \cup B_0 \cup B_1]$.

Proof. Suppose $\rho_1 = \pi_{A_1,\dots,A_m} \in \Pi_n^m$, $\rho_2 = \pi_{A'_1,\dots,A'_m} \in \Pi_n^m$. For every $j \in \{1, 2, \dots, m\}$ we have $A_j \subseteq A'_j$. It can be easily checked that

$$\rho_2(x_1, \dots, x_m, y_1, \dots, y_n) = \rho_1(x_1, \dots, x_m, y_1, \dots, y_n) \wedge \bigwedge_{i \in A'_j} \rho_W(x_j, y_i).$$

□

Lemma 3.25. *Suppose* $\rho_1, \rho_2 \in \Pi$, $\rho_2 \lesssim \rho_1$, *then* $\rho_2 \in [\{\rho_1\} \cup B_0 \cup B_1]$.

Proof. If $\sigma_1, \sigma_2 \in \Pi$, $\sigma_1 \simeq \sigma_2$, then σ_1 is obtained from σ_2 by a permutation of variables. Hence $\sigma_1 \in [\{\sigma_2\}]$ and $\sigma_2 \in [\{\sigma_1\}]$. Using this, Lemma 3.22, Lemma 3.23, and Lemma 3.24 we obtain that $\rho_2 \in [\{\rho_1\} \cup B_0 \cup B_1]$.

□

Theorem 3.26. *Suppose M is a clone,* $M \subseteq \mathrm{Pol}(\{\rho_{+1}, \{0, 1\}\})$ *and* $right \in M$. *Then* $M \in \Phi \cup \Upsilon \cup \{\mathbf{M}, \mathbf{C}\}$.

Proof. Let $S = \mathrm{Inv}(M)$. By Lemma 3.1, we have $B_0 \subseteq S$.

By Lemma 3.8, it follows that if $\rho_{=,012} \in S$, then $S = \mathrm{Inv}(right)$ and $M = [\{right\}] = \mathbf{aW} \in \Phi$. Suppose $\rho_{=,012} \notin S$.

Suppose $\rho_{=,01} \in S$. It follows from Lemma 3.9 that $[\{\rho_{=,01}, \rho_{+1}\}]$ contains every predicate $\rho \in \mathrm{Inv}(right)$ such that $\rho_{=,012} \notin [\{\rho, \rho_{=,01}, \rho_{+1}\}]$. Therefore, $S = [\{\rho_{=,01}, \rho_{+1}\}]$ and $M = \mathbf{aQ} \in \Phi$. Suppose $\rho_{=,01} \notin S$.

Suppose $\rho_{\to,2} \in S$, then by Lemma 3.19 we get $B_2 \subseteq S$. By Lemma 3.17, every predicate from S is equivalent to κ_n with respect to $B_0 \cup B_2$ for some $n \geq 0$. Let n_0 be the maximal number such that $\kappa_{n_0} \in S$. Hence, we have $S = [\{\kappa_{n_0}\} \cup B_0 \cup B_2]$. If $n_0 = 0$, then $M = \mathbf{aP} \in \Phi$, if $n_0 = 1$, then $M = \mathbf{aPN} \in \Phi$, if $n_0 \geq 2$, then $M = \mathbf{aP_{n_0-1}} \in \Phi$. If n_0 does not exist then $S = [\{\kappa_n \mid n \in \mathbb{N}_0\} \cup B_0 \cup B_2]$ and $M = \mathbf{aP}_\infty \in \Phi$.

Suppose $\rho_{\rightarrow,2} \notin S$. By Lemma 3.21 we have

$$S \cap \widetilde{R}_3 \subseteq \mathrm{Shift}(B_0 \cup B_1 \cup \Pi).$$

Suppose $\rho_W \in S$. By Lemma 3.5, we have $B_1 \subseteq S$. Let $S_\pi = S \cap \Pi$. By Lemma 3.25, S_π is a downset. By Lemma 3.11, $\pi_{\varnothing,\varnothing,\varnothing} \in S_\pi$. Hence $S_\pi \neq \varnothing$. Therefore $S \cap \widetilde{R}_3 = \mathrm{Shift}(S_\pi \cup B_0 \cup B_1)$ and $M = \mathrm{Clone}(S_\pi) \in \Upsilon$.

Suppose $\rho_W \notin S$, $\rho_{\vee,2} \in S$. By Lemma 3.12 we have $S \cap \Pi \subseteq \Pi_0$, where $\Pi_0 = \{\rho_{\vee,i} \mid i \geq 3\}$. Let n_0 be the maximal number such that $\rho_{\vee,n_0} \in S$. If n_0 does not exist then put $n_0 = \infty$. Obviously $\rho_{\vee,i}$ can be obtained from $\rho_{\vee,i+1}$ by identification of variables. Hence $\rho_{\vee,i} \in [\{\rho_{\vee,i+1}\}]$ for every i. Using Lemma 3.10, we get $\rho_{\leq} \in [\{\rho_N, \rho_{+1}\}]$. Therefore, only the following cases are possible:

1. $\rho_N \in S$, hence $M = \mathbf{a_{n_0}N} \in \Phi$,
2. $\rho_N \notin S$, $\rho_{\leq} \in S$, hence $M = \mathbf{a_{n_0}M} \in \Phi$,
3. $\rho_{\leq} \notin S$, hence $M = \mathbf{a_{n_0}} \in \Phi$.

Suppose $\rho_{\vee,2} \notin S$, then using Lemma 3.10 we get either $S \cap \widetilde{R}_3 = \mathrm{Shift}(B_0)$ and $M = \mathbf{C} \in \Theta$, or $S \cap \widetilde{R}_3 = \mathrm{Shift}(B_0 \cup \{\rho_{\leq}\})$ and $M = \mathbf{M} \in \Theta$. This completes the proof.

□

4 PROOF OF THE MAIN STATEMENTS AND THEOREMS

4.1 Correctness of the description of the lattice

Let

$$B_3 = \left\{ \{0\}, \{0,1\}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \rho_{+1}, \rho_{\leq}, \sigma_3^{=}, \rho_{\vee,2}, \rho_N, \rho_W \right\}.$$

Obviously, $B_3 \subseteq B_0 \cup B_1$.

The proof of the following theorem is rather cumbersome and complicated. It does not contain any interesting idea. We just check carefully all conditions from the definition of essentially closed set. That is why, we omit the proof and refer the reader to [12].

Theorem 4.1. *Suppose* $F \in \widetilde{\Pi}$, *then* $\mathrm{Shift}(F \cup B_3)$ *is essentially closed.*

Lemma 4.2. *Suppose* $\pi_{A_1,\ldots,A_m} \in \Pi_n^m$, $m + n \geq 3$, *then* 0^{m+n} *is a unique essential word for* $\pi_{A_1,\ldots,A_m}$.

Proof. Assume that $\alpha \neq 0^{m+n}$ and α is essential for $\pi_{A_1,\dots,A_m}$. Let

$$\rho_0(x_1,\dots,x_m,y_1,\dots,y_n) = \left(\bigwedge_{j\in A_i} \rho_W(x_i,y_j)\right) \wedge \left(\bigwedge_{i} x_i \in \{0,1\}\right).$$

Then $\pi_{A_1,\dots,A_m}(\beta) = \rho_0(\beta)$ for every $\beta \neq 0^{m+n}$. Then α is essential for ρ_0. But the predicate ρ_0 is not essential. This contradiction concludes the proof.

□

Lemma 4.3. *Suppose* $F \in \widetilde{\Pi}$, $\rho \in \mathrm{Shift}(F) \cap \Pi$, *then* $\rho \in F$.

Proof. Suppose $\rho \in \mathrm{Shift}(\rho')$, where $\rho' \in \Pi_n^m \cap F$. By Lemma 4.2, 0^{m+n} is a unique essential word for ρ and ρ'. Hence, there exists a permutation $\sigma : \{1,2,\dots,m+n\} \to \{1,2,\dots,m+n\}$ such that

$$\rho(z_1,z_2,\dots,z_{m+n}) = \rho'(z_{\sigma(1)},z_{\sigma(2)},\dots,z_{\sigma(m+n)}).$$

Since $\rho \in \Pi$, we obtain that $\mathrm{VarValues}(\rho,i) = \{0,1\}$ for $i \leq m$, and $\mathrm{VarValues}(\rho,i) = E_3$ for $i > m$. Therefore, $\sigma(i) \leq m$ for every $i \leq m$. Let

$$\rho_0(z_1,z_2,\dots,z_{m+n}) = \rho'(z_1,\dots,z_m,z_{\sigma(m+1)},z_{\sigma(m+2)},\dots,z_{\sigma(m+n)}).$$

It can be easily checked that

$$\rho(z_1,z_2,\dots,z_{m+n}) = \rho_0(z_{\sigma(1)},z_{\sigma(2)},\dots,z_{\sigma(m)},z_{m+1},\dots,z_{m+n}).$$

Then $\rho \lesssim^2 \rho_0 \simeq \rho'$. Hence, $\rho \lesssim \rho'$ and $\rho \in F$. This completes the proof.

□

Theorem 4.4. *Suppose* $F_1, F_2 \in \widetilde{\Pi}$, *then*

$$\mathrm{Clone}(F_1) \subseteq \mathrm{Clone}(F_2) \Longleftrightarrow F_1 \supseteq F_2.$$

Proof. If $F_1 \supseteq F_2$, then obviously $\mathrm{Clone}(F_1) \subseteq \mathrm{Clone}(F_2)$.

Suppose $\mathrm{Clone}(F_1) \subseteq \mathrm{Clone}(F_2)$, then

$$[F_1 \cup \{\rho_W, \rho_{+1}\}] \supseteq [F_2 \cup \{\rho_W, \rho_{+1}\}].$$

By Lemma 3.5 we have $B_3 \subseteq [\{\rho_W, \rho_{+1}\}]$. Since $\{\rho_W, \rho_{+1}\} \subseteq B_3$ we get $[\{\rho_W, \rho_{+1}\}] = [B_3]$ and thus $[F_1 \cup B_3] \supseteq [F_2 \cup B_3]$. So, we have

$$[F_1 \cup B_3] \cap \widetilde{R}_3 \supseteq [F_2 \cup B_3] \cap \widetilde{R}_3.$$

By Theorem 4.1, $\mathrm{Shift}(F_1 \cup B_3)$ and $\mathrm{Shift}(F_2 \cup B_3)$ are essentially closed. Then it follows from Lemma 2.2 and Theorem 2.8 that

$$[F_1 \cup B_3] \cap \widetilde{R}_3 = \mathrm{Shift}(F_1 \cup B_3),\ [F_2 \cup B_3] \cap \widetilde{R}_3 = \mathrm{Shift}(F_2 \cup B_3).$$

Hence $\mathrm{Shift}(F_1 \cup B_3) \supseteq \mathrm{Shift}(F_2 \cup B_3)$ and

$$\mathrm{Shift}(F_1 \cup B_3) \cap \Pi \supseteq \mathrm{Shift}(F_2 \cup B_3) \cap \Pi.$$

If $\rho \in F_2$, then $\rho \in \mathrm{Shift}(F_1) \cap \Pi$. By Lemma 4.3 we have $\rho \in F_1$. Hence, $F_1 \supseteq F_2$ and the theorem is proved.

□

Theorem 4.5 ([9, 12])**.** *Suppose M is a clone in P_3, $M \subseteq \mathrm{Pol}(\rho_{+1})$, and $M \not\subseteq \mathrm{Pol}(\{0,1\})$. Then $M \in \Theta$.*

Theorem 4.6 ([9, 12])**.** *Suppose M is a clone in P_3, $right, left \notin M$, and $M \subseteq \mathrm{Pol}(\{\rho_{+1}, \{0,1\}\})$. Then $M \in \Theta$.*

Theorem 4.7. *$\Theta \cup \Phi \cup \Upsilon$ is the set of all clones M such that $M \subseteq \mathrm{Pol}\,(\rho_{+1})$.*

Proof. Suppose $M \subseteq \mathrm{Pol}(\rho_{+1})$. We shall prove that $M \in \Upsilon \cup \Theta \cup \Phi$. Suppose $S = \mathrm{Inv}(M)$.

If $\{0,1\} \notin S$, then using Theorem 4.5, we get $M \in \Theta$.

Suppose $\{0,1\} \in S$. If $right \in M$ or $left \in M$, then by Theorem 3.26 we have $M \in \Phi \cup \Upsilon \cup \{\mathbf{M}, \mathbf{C}\}$.

If $right \notin M$, $left \notin M$ then it follows from Theorem 4.6 that $M \in \Theta$. This completes the proof.

□

4.2 Pairwise inclusion of clones into each other

Theorem 4.8. *Suppose $t \geq 3$, $F \in \widetilde{\Pi}$, then $\mathrm{Clone}(F) \subset \mathbf{a_t N}$ iff either $t = 3$ or $F \not\subseteq \Pi^{t-1}$.*

Proof. First, let us show that $\mathrm{Clone}(F) \neq \mathbf{a_t N}$. It is sufficient to check that the function $s_N \in \mathbf{a_t N}$ and s_N does not preserve ρ_W.

Suppose $\mathrm{Clone}(F) \subset \mathbf{a_t N}$. Since $\mathrm{Shift}(F \cup B_3)$ is essentially closed, we see that $\rho_{\vee,t} \in F$. Hence, either $t = 3$, or $F \not\subseteq \Pi^{t-1}$.

Let us prove the sufficiency. The proof follows from Lemma 3.11 for $t = 3$. Suppose $t \geq 4$, $\rho \in F \setminus \Pi^{t-1}$, then $\rho \in \Pi^m_n$, where $m + n \geq t$. It can be easily checked that $\rho_{\vee,t} \lesssim^2 \rho_{\vee,m+n} \lesssim^1 \rho$. Therefore, $\rho_{\vee,t} \lesssim \rho$ and $\rho_{\vee,t} \in F$. By the definition of $\mathbf{a_t N}$, we have $\mathrm{Clone}(F) \subset \mathbf{a_t N}$. This completes the proof.

□

Lemma 4.9. *Suppose $n \in \mathbb{N}$, then $g_n \in \mathbf{aP_n}$, $g_n \notin \mathbf{aP_{n+1}}$.*

Proof. It can be easily checked that $g_1 \in \mathrm{Pol}(\rho_W)$ and $g_n \in \mathrm{Pol}(\rho_{\to,2})$ for every $n \in \{1, 2, \ldots\}$. Hence $g_1 \in \mathbf{aP_1}$. Let us prove that $g_n \in \mathrm{Pol}(\pi_{\{1,\ldots,n\}})$ for $n \geq 2$. Assume the converse. Then there exist $\alpha_1, \ldots, \alpha_{n+2} \in \pi_{\{1,\ldots,n\}}$ such that

$$g_n(\alpha_1, \alpha_2, \ldots, \alpha_{n+2}) = \beta \notin \pi_{\{1,\ldots,n\}}.$$

Since g_n preserves $\{0, 1\}$ and $\beta \notin \pi_{\{1,\ldots,n\}}$, we have $\beta(1) = 0$. Hence, $\alpha_i(1) = 0$ for every $i \leq n+1$. Then it is easy to show that $\alpha_i \in \{0,1\}^{n+1}$ for every $i \leq n+1$. Therefore, $\beta \in \{0,1\}^{n+1}$. By the definition of $\pi_{\{1,\ldots,n\}}$, α_i contains 1 for every $i \leq n+1$. Hence, there exist $j \in \{2, 3, \ldots, n+1\}$ and $i_1, i_2 \in \{1, 2, \ldots, n+1\}$ such that $i_1 \neq i_2$ and $\alpha_{i_1}(j) = \alpha_{i_2}(j) = 1$. Therefore, $\beta(j) = 1$ and $\beta \in \pi_{\{1,\ldots,n\}}$. This contradiction proves that $g_n \in \mathrm{Pol}(\pi_{\{1,\ldots,n\}})$ for every $n \geq 2$.

Let us prove that $g_n \notin \mathrm{Pol}(\pi_{\{1,2,\ldots,n+1\}})$. Let $\alpha_i = 0^i 1 0^{n+1-i}$ for every $i \in \{1, 2, \ldots, n+1\}$, $\alpha_{n+2} = 12^{n+1}$. Obviously, $\alpha_i \in \pi_{\{1,2,\ldots,n+1\}}$ for every $i \in \{1, 2, \ldots, n+2\}$, and

$$g_n(\alpha_1, \ldots, \alpha_{n+2}) = 0^{n+2} \notin \pi_{\{1,2,\ldots,n+1\}}.$$

This completes the proof. □

Lemma 4.10. *Suppose $t \geq 1$, then $\Pi_t \subseteq [\{\rho_{+1}, \rho_{\to,2}, \rho_W, \pi_{\{1,2,\ldots,t\}}]$.*

Proof. Suppose $\pi_{A_1,\ldots,A_m} \in \Pi_n^m$, where $n \leq t$. If $n = 0$, then the lemma follows from Lemma 3.19. Suppose $n \geq 1$. By Lemma 3.13 and Lemma 3.19 we have

$$\pi_{\{1,2\ldots,n\}}, \rho_{\to,m} \in [\{\rho_{+1}, \rho_{\to,2}, \rho_W, \pi_{\{1,2,\ldots,t\}}\}].$$

It can be easily checked that

$$\pi_{A_1,\ldots,A_m}(x_1, \ldots, x_m, y_1, \ldots, y_n) =$$
$$\exists z\ \rho_{\to,m}(x_1, \ldots, x_m, z) \wedge \pi_{\{1,2,\ldots,n\}}(z, y_1, \ldots, y_n) \wedge (\bigwedge_{i \in A_j} \rho_W(x_j, y_i)).$$

This completes the proof. □

Lemma 4.11. $f_\pi^\infty \in \mathrm{Pol}(\Pi \cup B_3)$.

Proof. It is easy to check manually that $f_\pi^\infty \in \mathrm{Pol}(B_3)$. Let $\pi_{A_1,\dots,A_m} \in \Pi_n^m$. Assume that $f_\pi^\infty \notin \mathrm{Pol}(\pi_{A_1,\dots,A_m})$. Let

$$\rho_0(x_1,\dots,x_m,y_1,\dots,y_n) = \left(\bigwedge_{j\in A_i} \rho_W(x_i,y_j)\right) \wedge \left(\bigwedge_i x_i \in \{0,1\}\right).$$

Obviously, $\pi_{A_1,\dots,A_m}(\gamma) = \rho_0(\gamma)$ for every $\gamma \in \{0,1\}^{m+n} \setminus \{0^{m+n}\}$. Since $f_\pi^\infty \in \mathrm{Pol}(B_3)$, we have $f_\pi^\infty \in \mathrm{Pol}(\rho_0)$. Suppose $\alpha_1,\alpha_2,\alpha_3 \in \pi_{A_1,\dots,A_m}$, $f_\pi^\infty(\alpha_1,\alpha_2,\alpha_3) = \beta \notin \pi_{A_1,\dots,A_m}$. Since $f_\pi^\infty(\alpha_1,\alpha_2,\alpha_3) \in \rho_0$, we get $\beta = 0^{m+n}$.

Since $\alpha_1 \in \pi_{A_1,\dots,A_m}$, there exists j such that $\alpha_1(j) = 1$. By the definition of f_π^∞ we obtain that $\beta(j) \in \{1,2\}$. This contradiction completes the proof.

□

Theorem 4.12. *Suppose* $F \in \widetilde{\Pi}$, *then* $\mathbf{aP_t} \subset \mathrm{Clone}(F)$ *iff* $F \subseteq \Pi_t$.

Proof. Let

$$S = \mathrm{Inv}(\mathbf{aP_t}) = \left[\left\{\rho_{+1}, \rho_{\to,2}, \rho_W, \pi_{\{1,2\dots,t\}}\right\}\right].$$

Let us prove the necessity. Suppose $\mathbf{aP_t} \subset \mathrm{Clone}(F)$, then $F \subseteq S$. Assume that $F \not\subseteq \Pi_t$, then there exists $\rho \in \Pi_n^m \cap F$ such that $n > t$. Obviously,

$$\pi_{\{1,2,\dots,n\}}(x,y_1,\dots,y_n) = \rho(x,\dots,x,y_1,\dots,y_n).$$

Therefore, $\pi_{\{1,2,\dots,n\}} \in F \subseteq S$. Hence $\mathbf{aP_n} = \mathbf{aP_t}$, where $n > t$. This contradicts to Lemma 4.9.

Let us prove the sufficiency. By Lemma 4.10 we have $\Pi_t \subseteq S$. Hence, $F \subseteq \Pi_t \subseteq S$ and $\mathbf{aP_t} \subseteq \mathrm{Clone}(F)$.

To complete the proof we have to show that $\mathbf{aP_t} \neq \mathrm{Clone}(F)$. It follows from Lemma 4.11 that $f_\pi^\infty \in \mathrm{Clone}(F)$. It is easy to check that f_π^∞ does not preserve $\rho_{\to,2}$, hence $f_\pi^\infty \notin \mathbf{aP_t}$. This completes the proof.

□

Theorem 4.13. *Suppose* $F \in \widetilde{\Pi}$, *then* $\mathbf{aP}_\infty \subset \mathrm{Clone}(F)$.

Proof. By Lemma 4.10 we obtain $\Pi \subset \mathrm{Inv}(\mathbf{aP}_\infty)$. Hence, $F \subseteq \mathrm{Inv}(\mathbf{aP}_\infty)$ and $\mathbf{aP}_\infty \subseteq \mathrm{Clone}(F)$.

To complete the proof we have to show that $\mathbf{aP}_\infty \neq \mathrm{Clone}(F)$. By Lemma 4.11 we have $f_\pi^\infty \in \mathrm{Clone}(F)$. It is easy to check that f_π^∞ does not preserve $\rho_{\to,2}$, hence $f_\pi^\infty \notin \mathbf{aP}_\infty$. This completes the proof.

□

4.3 Bases of clones

Lemma 4.14. *Suppose $l \geq 3$, then $f_\pi^l \in \mathrm{Pol}(\Pi^l \cup B_3)$, $f_\pi^l \notin \mathrm{Pol}(\rho_{\vee,l+1})$,*

Proof. It can be easily shown that $f_\pi^l \in \mathrm{Pol}(B_3)$. Assume that $f_\pi^l \notin \mathrm{Pol}(\rho)$ for some $\rho = \pi_{A_1,\ldots,A_m} \in \Pi_n^m$, where $m + n \leq l$. Let

$$\rho_0(x_1,\ldots,x_m,y_1,\ldots,y_n) = \left(\bigwedge_{j\in A_i} \rho_W(x_i,y_j)\right) \wedge \left(\bigwedge_i x_i \in \{0,1\}\right).$$

Since $f_\pi^l \in \mathrm{Pol}(B_3)$, we have $f_\pi^l \in \mathrm{Pol}(\rho_0)$.

Suppose $\alpha_1,\alpha_2,\ldots,\alpha_{l+1} \in \rho$, $f_\pi^l(\alpha_1,\alpha_2,\ldots,\alpha_{l+1}) = \beta \notin \rho$. Since $f_\pi^l \in \mathrm{Pol}(\rho_0)$ and $\rho(\gamma) = \rho_0(\gamma)$ for every $\gamma \neq 0^{m+n}$, we obtain $\beta = 0^{m+n}$.

Every word $\alpha_1,\ldots,\alpha_{l+1}$ contains at least one 1. Let i be the minimal number such that $\alpha_{j_1}(i) = \alpha_{j_2}(i) = 1$ for some j_1, j_2, $j_1 \neq j_2$. Since $m + n < l + 1$, this number exists.

If $i \leq m$, then obviously $\beta(i) = 1$. This contradicts the assumption.

If $i > m$, then there exists j_3, j_4, $j_3 \neq j_4$ such that $\alpha_{j_3}(i) = \alpha_{j_4}(i) = 2$. By the definition of Π there exists $i' \leq m$ such that $(i - m) \in A_{i'}$. Hence $\alpha_{j_3}(i') = \alpha_{j_4}(i') = 1$. This contradicts the assumption about the minimality of i.

To complete the proof we need to show that f_π^l does not preserve $\rho_{\vee,l+1}$. Let $\alpha_i = 0^{i-1}10^{l+1-i} \in \rho_{\vee,l+1}$ for $i \in \{1,2,\ldots,l+1\}$. It can be easily checked that $f_\pi^l(\alpha_1,\alpha_2,\ldots,\alpha_{l+1}) = 0^{l+1} \notin \rho_{\vee,l+1}$. Therefore, f_π^l does not preserve $\rho_{\vee,l+1}$.

□

Lemma 4.15. *Suppose $n \geq 2$, then $r_3 \in \mathrm{Pol}(\pi_{\{1,2,\ldots,n\}})$.*

Proof. Assume the converse. Then there exist $\alpha_1,\alpha_2,\alpha_3 \in \pi_{\{1,2,\ldots,n\}}$ such that $r_3(\alpha_1,\alpha_2,\alpha_3) = \beta \notin \pi_{\{1,2,\ldots,n\}}$. Since r_3 preserves $\{0,1\}$, we see that $\beta(1) \in \{0,1\}$. Therefore we get $\beta(1) = 0$, $\alpha_1(1) = \alpha_2(1) = 0$ and $\alpha_1,\alpha_2 \in \{0,1\}^{n+1}$. Hence, $\beta \in \{0,1\}^{n+1}$. By the definition of $\pi_{\{1,2,\ldots,n\}}$ there exists $i \geq 2$ such that $\alpha_1(i) = 1$. It can be easily checked that $\beta(i) = 1$. This completes the proof.

□

Theorem 4.16. *The clones of the class Θ have the following bases:*

$$\mathbf{S} = [\{x+1, right\}] = [\{x+1, left\}],$$

$$\mathbf{S_0} = [\{2x+2y, right\}] = [\{2x+2y, left\}],$$

$$\mathbf{SL} = [\{2x+2y, x+1\}] = [\{2x+2y+1\}], \ \ \mathbf{1S} = [\{x+1\}],$$

$$\mathbf{SL_0} = [\{2x+2y\}], \ \ \mathbf{T} = [\{2x+2y, ps\}],$$

$$\mathbf{C} = [\{plus, right\}] = [\{plus, left\}],$$

$$\mathbf{D} = [\{plus, m_0\}] = [\{plus_0, m\}] = [\{plus, m, ps_0\}],$$

$$\mathbf{M} = [\{right, left\}], \ \ \mathbf{DM} = [\{m, ps_0\}] = [\{m_0, ps\}],$$

$$\mathbf{DN} = [\{m_0\}], \ \ \mathbf{TD} = [\{m, plus\}], \ \ \mathbf{TM} = [\{ps, m\}],$$

$$\mathbf{TN} = [\{m\}], \ \ \mathbf{L} = [\{plus, ps_0\}] = [\{plus_0\}],$$

$$\mathbf{TL} = [\{plus\}], \ \ \mathbf{C_2} = [\{ps_0\}], \ \ \mathbf{TC_2} = [\{ps\}], \ \ \mathbf{J_3} = [\{x\}].$$

Theorem 4.17. *The clones of the class Φ have the following bases:*

$$\mathbf{a_2} = [\{f_0^\infty, m\}], \ \ \mathbf{a_2M} = [\{ps, right, m\}], \ \ \mathbf{a_2N} = [\{m, right\}].$$

For $n \geq 3$

$$\mathbf{a_n} = [\{f_0^\infty, f_\pi^n\}] = [\{f_0^\infty, f_0^n\}],$$

$$\mathbf{a_nM} = [\{f_\pi^n, ps\}] = [\{f_0^n\}], \ \ \mathbf{a_nN} = [\{f_\pi^n, s_N\}],$$

$$\mathbf{a}_\infty = [\{f_0^\infty\}], \ \ \mathbf{a}_\infty\mathbf{M} = [\{f_\pi^\infty, ps\}], \ \ \mathbf{a}_\infty\mathbf{N} = [\{f_\pi^\infty, s_N\}].$$

For $n \geq 1$

$$\mathbf{aP} = [\{right, ps\}], \ \ \mathbf{aPN} = [\{s_N\}], \ \ \mathbf{aP_n} = [\{g_n\}],$$

$$\mathbf{aP}_\infty = [\{r_3\}], \ \ \mathbf{aQ} = [\{r_4\}], \ \ \mathbf{aW} = [\{right\}].$$

Theorem 4.18. *For $n \geq 3$ and $m \geq 1$ clones of the class Υ have the following bases:*

$$\mathbf{a}_\infty\pi_\infty = [\{f_\pi^\infty\}], \ \ \mathbf{a_n}\pi_\infty = [\{f_\pi^n\}],$$

$$\mathbf{a}_\infty\pi_\mathbf{0} = [\{s_0\}], \ \ \mathbf{a_n}\pi_\mathbf{0} = [\{s_0, f_\pi^n\}],$$

$$\mathrm{Clone}(\Pi_m) = [\{g_m, f_\pi^\infty\}], \ \ \mathrm{Clone}(\Pi_1 \cap \Pi^n) = [\{g_1, f_\pi^n\}].$$

Theorem 4.16 can be easily checked manually. Moreover bases for clones from the class Θ were already found in [9]. The sketched proof of Theorem 4.17 and Theorem 4.18 is below.

Proof. The main points of the proof are listed below:

1. $f_\pi^l \in \mathrm{Pol}(\Pi^l \cup B_3)$, $f_\pi^l \notin \mathrm{Pol}(\rho_{\vee,l+1})$ for $l \geq 3$ (by Lemma 4.14);

2. $f_0^\infty \in \mathrm{Pol}(\rho_{\vee,n})$ for every $n \geq 3$ (it follows from the properties of the function $x \vee y\bar{z} \in P_2$);

3. $f_0^n \in \mathrm{Pol}(\rho_{\vee,n})$, $f_0^n \notin \mathrm{Pol}(\rho_{\vee,n+1})$ for every $n \geq 3$ (it follows from the properties of the function $h_n^* \in P_2$);

4. $f_\pi^\infty \in \mathrm{Pol}(\Pi \cup B_3)$ (by Lemma 4.11);

5. $g_n \in \mathrm{Pol}(\pi_{\{1,2,\dots,n\}})$, $g_n \notin \mathrm{Pol}(\pi_{\{1,2,\dots,n,n+1\}})$ for every $n \geq 2$ (by Lemma 4.9);

6. r_3 preserves $\pi_{\{1,2,\dots,n\}}$ for every $n \geq 2$ (by Lemma 4.15).

To complete the proof we need to check that every clone M contains the corresponding set of function B, B is not a subset of any clone $M' \subset M$. Moreover, we have to check that for every $f \in B$ there exists a clone $M' \subset M$ such that $B \setminus \{f\} \subseteq M'$.

□

We will need the definition of B_Π and Bound from Section 1.

Theorem 4.19. $\mathrm{Bound} : \widetilde{\Pi} \to B_\Pi$ *is a bijective mapping.*

Proof. It is easy to check that if $\rho, \sigma \in \Pi$ and $\rho \lesssim \sigma$ then $\mathrm{ar}(\rho) \leq \mathrm{ar}(\sigma)$. This means that for every predicate $\rho \in \Pi$ the set $\{\sigma \mid \sigma \lesssim \rho\}$ is finite. Then there are no infinite descending chains in E_Π. Hence there is a bijective correspondence between antichains of E_Π and upsets of E_Π.

There is also a natural bijective correspondence between downsets and upsets: the complement of an upset is a downset and vice versa. Combining these two, we get a bijective correspondence between downsets and antichains, where the antichain corresponding to a downset is just the set of minimal elements of the complement of this downset. The map Bound describe exactly this correspondence, with two minor modifications:

1. we work with the quasiordered set Π but not with the poset E_Π, hence we have to take into account the corresponding equivalence relation.

2. the empty downset is excluded as well as the corresponding antichain (namely, the one-element antichain containing only the bottom element).

□

Theorem 4.20. *Suppose* $M \subset \mathrm{Clone}(F)$, $F \in \widetilde{\Pi} \setminus \{\Pi, \Pi_0, \Pi_1, \Pi_2, \Pi_3, \ldots\}$, *and* $g : \mathrm{Bound}(F) \to M$ *is a bijective mapping such that for every* $\widehat{\rho} \in \mathrm{Bound}(F)$ *we have*

$$g(\widehat{\rho}) \in \mathrm{Clone}(F_{\widehat{\rho}}) \setminus \mathrm{Clone}(F_0).$$

Then M *is a basis for* $\mathrm{Clone}(F)$.

Proof. First we shall show that $[M] \in \Upsilon$. Let n be the minimal number such that $\Pi_n \not\subseteq F$. Since $F \neq \Pi$, this number exists. Let $\rho' \in \Pi_n \setminus F$. By the definition, there exists $\rho \lesssim \rho'$ such that $\widehat{\rho} \in \mathrm{Bound}(F)$. It can easily be checked that $\rho \in \Pi_n \setminus \Pi_{n-1}$. It follows from the condition that $g(\widehat{\rho}) \notin \mathrm{Pol}(\rho)$. Hence, $M \not\subseteq \mathrm{Pol}(\Pi_n)$ and $M \not\subseteq \mathrm{Clone}(\Pi_n)$. By Theorem 4.12 we obtain that $\mathbf{aP_n} \subseteq \mathrm{Clone}(\Pi_n)$. Hence, $M \not\subseteq \mathbf{aP_n}$.

Since $F \neq \Pi_{n-1}$, we see that $F \not\subseteq \Pi_{n-1}$. By the condition of the theorem, $M \subseteq \mathrm{Clone}(F)$. Then using Theorem 4.12 we obtain that $[M] \neq \mathbf{aP_t}$ for every $t < n$. It follows from the description of the lattice that $[M] \in \Upsilon$.

Thus there exists $G \in \widetilde{\Pi}$ such that $[M] = \mathrm{Clone}(G)$. Let us show that $F = G$. Assume the converse, then $F \subset G$. Let δ be a minimal predicate in $G \setminus F$ with respect to $\lesssim$. Since G is a downset, we have $\widehat{\delta} \in \mathrm{Bound}(F)$. Then by definition, $g(\widehat{\delta}) \notin \mathrm{Pol}(\delta)$. Therefore, $g(\widehat{\delta}) \notin \mathrm{Clone}(G)$. This contradiction proves that $[M] = \mathrm{Clone}(F)$.

Let us prove that M is a basis. Assume the converse. Then there exists $M' \subset M$ such that $[M'] = \mathrm{Clone}(F)$. Suppose $f \in M \setminus M'$, $\widehat{\delta} = g^{-1}(f)$. It can be easily checked that $F \cup \widetilde{\delta}$ and

$$M' \subseteq \mathrm{Clone}(F \cup \widetilde{\delta}).$$

Hence, $[M'] \neq \mathrm{Clone}(F)$. This completes the proof.

□

Corollary 4.21. *Suppose* $M \in \Theta \cup \Phi \cup \Upsilon$, *then* M *has a basis.*

Corollary 4.22. *Suppose* $F \in \widetilde{\Pi}$, *then* $\mathrm{Clone}(F)$ *is finitely generated iff* $\mathrm{Bound}(F)$ *is finite.*

The proof of these two corollaries is below.

Proof. Suppose that $\mathrm{Clone}(F) \in \Upsilon$. Let us consider two cases. Suppose $F \in \{\Pi, \Pi_0, \Pi_1, \Pi_2, \ldots\}$, then by Theorem 4.18 $\mathrm{Clone}(F)$ is finitely generated. It can be checked that $\mathrm{Bound}(\Pi) = \varnothing$, $\mathrm{Bound}(\Pi_0) = \{\pi_{\{1\},\{1\}}\}$, $\mathrm{Bound}(\Pi_i) = \{\pi_{\{1,2,\ldots,i+1\}}\}$ for every $i \in \{1, 2, 3, \ldots\}$. This completes the case.

Suppose $F \notin \{\Pi, \Pi_0, \Pi_1, \Pi_2, \ldots\}$. Let us define $M \subseteq \mathrm{Clone}(F)$. For every $\widehat{\rho} \in \mathrm{Bound}(F)$ we select a function from $\mathrm{Clone}(F_{\widehat{\rho}}) \setminus \mathrm{Clone}(F_0)$. It follows from Theorem 4.20 that M is a basis in $\mathrm{Clone}(F)$. If $\mathrm{Bound}(F)$ is finite, then M is a finite basis in $\mathrm{Clone}(F)$.

Suppose $\mathrm{Bound}(F)$ is infinite, then M is an infinite basis. Assume that there exists a finite basis M_0 in $\mathrm{Clone}(F)$. Then, there exists a finite set $M' \subset M$ such that $M_0 \subseteq [M']$. Then $\mathrm{Clone}(F) = [M']$. This contradicts the fact that M is a basis.

□

Corollary 4.23. *Suppose $F \in \widetilde{\Pi}$, $|F| < \infty$, then* $\mathrm{Clone}(F)$ *is finitely generated.*

Proof. Let us show that $\mathrm{Bound}(F)$ is finite if F is finite. It can be checked that if $\rho \lesssim \sigma$ and $\mathrm{ar}(\rho)+1 < \mathrm{ar}(\sigma)$ then there exists σ_0 such that $\rho \lesssim \sigma_0 \lesssim \sigma$ and $\mathrm{ar}(\rho) + 1 = \mathrm{ar}(\sigma_0)$. Since F is finite, there exists $l \in \{3, 4, 5, \ldots\}$ such that $F \subseteq \Pi^l$. Hence $\mathrm{Bound}(F) \subseteq \Pi^{l+1}$ and $\mathrm{Bound}(F)$ is finite.

Then by Corollary 4.22 $\mathrm{Clone}(F)$ is finitely generated.

□

4.4 Relation degree

We will need the following well-known property of relation degree.

Lemma 4.24 ([5]). *Suppose $C_1 \supset C_2 \supset C_3 \supset \ldots$ is an infinite sequence of clones, $C_\infty = \bigcap\limits_i C_i$. Then $d(C_\infty) = \infty$.*

Theorem 4.25. *Suppose $M \in \Theta \cup \Phi$, then*

$$d(M) = \begin{cases} 2, & \text{if } M \in \{\mathbf{S}, \mathbf{S_0}, \mathbf{T}, \mathbf{C}, \mathbf{M}, \mathbf{D}, \mathbf{DM}, \mathbf{DN}, \mathbf{TD}, \mathbf{TM}, \\ & \mathbf{TN}, \mathbf{1S}, \mathbf{J_3}\}; \\ 3, & \text{if } M \in \{\mathbf{SL}, \mathbf{SL_0}, \mathbf{L}, \mathbf{TL}, \mathbf{C_2}, \mathbf{TC_2}, \mathbf{aP}, \mathbf{aPN}, \\ & \mathbf{aP_1}, \mathbf{aQ}, \mathbf{aW}, \mathbf{AP}, \mathbf{APN}, \mathbf{AP_1}, \mathbf{AQ}, \mathbf{AW}\}; \\ n, & \text{if } n \geq 2 \text{ and } M \in \{\mathbf{a_n}, \mathbf{a_n M}, \mathbf{a_n N}, \mathbf{A_n}, \mathbf{A_n M}, \mathbf{A_n N}\} \\ n+1, & \text{if } n \geq 2 \text{ and } M \in \{\mathbf{aP_n}, \mathbf{AP_n}\}; \\ \infty, & \text{if } M \in \{\mathbf{a_\infty}, \mathbf{a_\infty M}, \mathbf{a_\infty N}, \mathbf{aP_\infty}, \mathbf{A_\infty}, \mathbf{A_\infty M}, \\ & \mathbf{A_\infty N}, \mathbf{AP_\infty}\}; \end{cases}$$

Proof. For clones from the class Θ the proof follows from the description of the lattice. We refer the reader to [12] for more details. For clones from the set

$$\{\mathbf{aP}, \mathbf{aPN}, \mathbf{aP_1}, \mathbf{aQ}, \mathbf{aW}, \mathbf{AP}, \mathbf{APN}, \mathbf{AP_1}, \mathbf{AQ}, \mathbf{AW}\}$$

the theorem follows from the complete description of all essential predicates of arity 2 from $\mathrm{Inv}(right)$ in Lemma 3.4. Indeed, by Lemma 3.4 we have $\mathrm{Inv}(right) \cap \widetilde{R}_3 \subseteq \mathrm{Shift}(B_0 \cup Main)$. Hence

$$\mathrm{Inv}(right) \cap (\widetilde{R}_3^1 \cup \widetilde{R}_3^1) \subseteq \mathrm{Shift}(B_0 \cup B_1) \subseteq \mathrm{Inv}(\mathbf{a_3}\pi_\mathbf{0}).$$

So, every clone defined by such predicates contains $\mathbf{a_3}\pi_\mathbf{0}$, therefore these clones cannot be defined by predicates of arity 1 and 2.

For clones from the set

$$\{\mathbf{a_n}, \mathbf{a_n M}, \mathbf{a_n N}, \mathbf{A_n}, \mathbf{A_n M}, \mathbf{A_n N}\}$$

the theorem follows from the complete description of all essential predicates in the proof of Theorem 3.26.

For clones $\mathbf{aP_n}, \mathbf{AP_n}$, where $n \geq 1$, the theorem follows from the proof of Theorem 3.26.

For clones from the set

$$\{\mathbf{a}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}, \mathbf{aP}_\infty, \mathbf{A}_\infty, \mathbf{A}_\infty\mathbf{M}, \mathbf{A}_\infty\mathbf{N}, \mathbf{AP}_\infty\}$$

the theorem follows from Lemma 4.24.

□

Theorem 4.26. *Suppose* $F \in \widetilde{\Pi}$, $F \neq \{\pi_{\varnothing,\varnothing,\varnothing}\}$, *then*

$$d(\mathrm{Clone}(F)) = \begin{cases} \max\{m+n \mid \Pi_n^m \cap F \neq \varnothing\}, & \text{if } |F| < \infty; \\ \infty, & \text{otherwise.} \end{cases}$$

$d(\mathrm{Clone}(\{\pi_{\varnothing,\varnothing,\varnothing}\})) = 2.$

Proof. For $F = \{\pi_{\varnothing,\varnothing,\varnothing}\}$ the proof follows from Lemma 3.11 and the following equation:

$$\mathrm{Clone}(\{\pi_{\varnothing,\varnothing,\varnothing}\}) = \mathrm{Pol}(\{\rho_{+1}, \rho_W\}).$$

Assume that $F \neq \{\pi_{\varnothing,\varnothing,\varnothing}\}$, $\mathrm{Clone}(F) = \mathrm{Pol}(S)$, where $S \subseteq R_3$. By Lemma 2.5, it can be assumed that $S \subseteq \widetilde{R}_3$. It follows from Theorem 4.1 that the set $\mathrm{Shift}(F \cup B_3)$ is essentially closed. Hence $S \subseteq \mathrm{Shift}(F \cup B_3)$. Since $F \neq \{\pi_{\varnothing,\varnothing,\varnothing}\}$, we see that $S \cap \mathrm{Shift}(\Pi) \neq \varnothing$.

Let $F' = \bigcup\limits_{\rho \in \mathrm{Shift}(S) \cap \Pi} \{\sigma \in \Pi \mid \sigma \lesssim \rho\}$. Hence, we have

$$\mathrm{Clone}(F) = \mathrm{Pol}(S) \supseteq \mathrm{Pol}(\mathrm{Shift}(F' \cup B_3)) = \mathrm{Clone}(F').$$

Then, by Theorem 4.4, we get $F \subseteq F'$, and this implies

$$\max_{\rho\in F}(\mathrm{ar}(\rho)) \leq \max_{\rho\in F'}(\mathrm{ar}(\rho)) = \max_{\rho\in S}(\mathrm{ar}(\rho)).$$

This proves that

$$d(\mathrm{Clone}(F)) = \max_{\rho\in F}(\mathrm{ar}(\rho)) = \max\{m+n \mid \Pi_n^m \cap F \neq \varnothing\}.$$

□

4.5 Cardinalities of $\mathbb{L}_3^{\uparrow}(F)$ and $\mathbb{L}_3^{\downarrow}(F)$

Let us define a mapping $\zeta : \Pi \to \mathbb{N}_0 \times \mathbb{N}$. For $\pi_{A_1,\dots,A_m} \in \Pi_n^m$ we put $\zeta(\pi_{A_1,\dots,A_m}) = (a, b)$, where $a = n - \max_i\{|A_i|\}$, and b is the number of different sets A_j such that $|A_j| = \max_i\{|A_i|\}$. We define a linear order on the set $\mathbb{N}_0 \times \mathbb{N}$ in the following way:

$$(a_1, b_1) \leq (a_2, b_2) \iff (a_1 < a_2) \vee ((a_1 = a_2) \wedge (b_1 \geq b_2)).$$

The following two lemmas can easily be checked.

Lemma 4.27. *Suppose* $\rho_1, \rho_2 \in \Pi$, $\rho_1 \lesssim \rho_2$, *then* $\mathrm{ar}(\rho_1) \leq \mathrm{ar}(\rho_2)$.

Lemma 4.28. *Suppose* $\rho_1, \rho_2 \in \Pi$, $\rho_1 \lesssim \rho_2$, *then* $\zeta(\rho_1) \leq \zeta(\rho_2)$.

Theorem 4.29. *Suppose* $F \in \widetilde{\Pi}$, *then*

$$|\mathbb{L}_3^{\downarrow}(\mathrm{Clone}(F))| = \begin{cases} 2^{\aleph_0}, & \textit{if } F \neq \Pi; \\ 5, & \textit{if } F = \Pi. \end{cases}$$

Proof. Suppose $F = \Pi$, then the proof follows from the description of the lattice. Suppose $F \neq \Pi$, then there exists a predicate $\pi_{A_1,\dots,A_m} \in \Pi_n^m$ such that $\pi_{A_1,\dots,A_m} \notin F$. For $i, l \in \mathbb{N}$, we put

$$B_{i,l} = \{j \mid n < j \leq l+n, j \neq i+n\},$$

$$\rho_l = \pi_{A_1,\dots,A_m,B_{1,l},B_{2,l},\dots,B_{l,l}}.$$

It can be easily checked that

$$\pi_{A_1,\dots,A_m} \lesssim^2 \pi_{A_1,\dots,A_m,\varnothing,\dots,\varnothing} \lesssim^1 \pi_{A_1,\dots,A_m,B_{1,l},B_{2,l},\dots,B_{l,l}},$$

hence $\pi_{A_1,\dots,A_m} \lesssim \rho_l$. Since F is a downset, we have $\rho_l \notin F$ for every $l \in \mathbb{N}$.

Let $G = \{\rho_i \mid i \geq n+2\}$. For $i \geq n+2$ we have

$$\mathrm{ar}(\rho_i) = m + i + n + i, \;\; \zeta(\rho_i) = (n+1, i).$$

Hence, it follows from Lemma 4.27 and Lemma 4.28 that G consists of pairwise incomparable predicates. Suppose $G' \subseteq G$. Put

$$F_{G'} = F \cup \{\sigma \in \Pi \mid \exists \delta \in G' : \; \sigma \lesssim \delta\}.$$

It can be easily checked that $F_{G'}$ is a nonempty downset. By Corollary 1.2, if $G_1 \neq G_2$, then $F_{G_1} \neq F_{G_2}$. Let $M = \{\mathrm{Clone}(F_{G'}) \mid G' \subseteq G\}$. Obviously, the cardinality of M is continuum and $M \subseteq \mathbb{L}_3^{\downarrow}(\mathrm{Clone}(F))$. This completes the proof.

□

Theorem 4.30. *Suppose $M \in \Theta \cup \Phi$, then*

$$|\mathbb{L}_3^{\downarrow}(M)| \begin{cases} = \aleph_0, & \textit{if } M \in \{\mathbf{aP}, \mathbf{aPN}, \mathbf{aP_1}, \mathbf{aP_2}, \mathbf{aP_3}, \ldots, \\ & \mathbf{AP}, \mathbf{APN}, \mathbf{AP_1}, \mathbf{AP_2}, \mathbf{AP_3}, \ldots\}; \\ = 2^{\aleph_0}, & \textit{if } M \in \{\mathbf{S}, \mathbf{S_0}, \mathbf{C}, \mathbf{M}, \mathbf{a}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}, \\ & \mathbf{A}_\infty, \mathbf{A}_\infty\mathbf{M}, \mathbf{A}_\infty\mathbf{N}\} \\ & \textit{or } M \in \bigcup\limits_{n\geq 2} \{\mathbf{a_n}, \mathbf{a_n M}, \mathbf{a_n N}, \mathbf{A_n}, \mathbf{A_n M}, \mathbf{A_n N}\}; \\ < \infty, & \textit{otherwise.} \end{cases}$$

Proof. Suppose

$$M \in \{\mathbf{S}, \mathbf{S_0}, \mathbf{C}, \mathbf{M}, \mathbf{a}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}\} \cup (\bigcup_{n\geq 2} \{\mathbf{a_n}, \mathbf{a_n M}, \mathbf{a_n N}\}).$$

It follows from Theorem 4.29 that the cardinality of the set $\mathbb{L}_3^{\downarrow}(\mathbf{a}_\infty \pi_{\mathbf{0}})$ is continuum. Since $\mathbf{a}_\infty \pi_{\mathbf{0}} \subset M$, the cardinality of $\mathbb{L}_3^{\downarrow}(M)$ is also continuum.

The proof for other clones follows from the description of the lattice.

□

Theorem 4.31. *Suppose $F \in \widetilde{\Pi}$, then*

$$|\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))| \begin{cases} = 2^{\aleph_0}, & \textit{if } F \textit{ contains an infinite antichain;} \\ < \infty, & \textit{if } |F| < \infty; \\ = \aleph_0, & \textit{otherwise.} \end{cases}$$

Proof. Suppose F contains an infinite antichain G. Suppose $G' \subseteq G$ and G' is not empty. Put

$$F_{G'} = \{\sigma \in \Pi \mid \exists \delta \in G' :\ \sigma \lesssim \delta\}.$$

It is easy to prove that $F_{G'}$ is a nonempty downset. Obviously, if $G_1 \neq G_2$, then $F_{G_1} \neq F_{G_2}$. Let $M = \{\mathrm{Clone}(F_{G'}) \mid \varnothing \neq G' \subseteq G\}$. Then the cardinality of M is continuum and $M \subseteq \mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))$.

Suppose F is a finite set. Then it follows from the description of the lattice that $\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))$ is finite.

Suppose F is infinite, but F does not contain an infinite antichain. By Theorem 4.8, we have $\mathbf{a_n} \in \mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))$ for every $n \in \{2, 3, \ldots\}$. Therefore, $\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))$ is at least countable.

Let us prove that the set $\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F)) \cap \Upsilon$ is at most countable. Let $F' \in \widetilde{\Pi}$, $F' \subseteq F$. Put $G = \mathrm{Bound}(F')$, $\widetilde{F} = \{\widehat{\rho} \mid \rho \in F\}$. It can be easily checked that $G \subseteq \mathrm{Bound}(F) \cup \widetilde{F}$ and

$$G \cap \mathrm{Bound}(F) = \{\widehat{\rho} \in \mathrm{Bound}(F) \mid \forall \widehat{\sigma} \in G \cap \widetilde{F} :\ \neg(\widehat{\sigma} < \widehat{\rho})\}.$$

Hence, the set G is uniquely determined by the set $G \cap \widetilde{F}$. Since F does not contain an infinite antichain, the set $G \cap \widetilde{F}$ is finite. Therefore, every clone $\mathrm{Clone}(F')$ is defined by a finite set of predicates. Then the set $\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F)) \cap \Upsilon$ is at most countable. Since the class Θ is finite and the class Φ is countable, the set $\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))$ is countable.

□

Let us define a partial order on the set $\mathbb{N}_0^n$. We say that

$$(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n),$$

if for every $i \in \{1, 2, \ldots, n\}$ either $a_i = b_i = 0$, or $0 < a_i \leq b_i$.

Lemma 4.32. *Suppose $F \subseteq \mathbb{N}_0^n$, F is an antichain. Then F is finite.*

Proof. The proof is by induction on n. If $n = 1$ then the proof is trivial. Assume the converse. Suppose F is infinite. Let

$$sign(i) = \begin{cases} 1, & \text{if } i > 0; \\ 0, & \text{if } i = 0. \end{cases}.$$

To each tuple $(a_1, \ldots, a_n)$ from F assign a tuple $(sign(a_1), \ldots, sign(a_n))$ from $\{0, 1\}^n$. Since the set $\{0, 1\}^n$ is finite, there exists a tuple assigned to infinitely many tuples from F. Hence, without loss of generality it can be assumed that to each tuple from F we assign the same tuple from $\{0,1\}^n$.

Suppose $\alpha = (a_1, \ldots, a_n) \in F$. Then for every $\beta = (b_1, \ldots, b_n) \in F$ such that $\alpha \neq \beta$ we have

$$b_1 < a_1 \vee b_2 < a_2 \vee \ldots \vee b_n < a_n.$$

Hence, there exists $i \in \{1, 2, \ldots, n\}$ such that $b_i < a_i$ for infinitely many tuples $(b_1, \ldots, b_n)$ from F. Therefore, there exists $c < a_i$ such that $b_i = c$ for infinitely many tuples $(b_1, \ldots, b_n)$ from F. Let F_0 be the set of all tuples $(b_1, \ldots, b_n) \in F$ such that $b_i = c$. Obviously, by removing i-th element from every tuple from F_0 we obtain an infinite antichain from $\mathbb{N}_0^{n-1}$. This contradicts the inductive assumption.

□

Lemma 4.33. *Suppose $F \subseteq \Pi_n$, F is an antichain. Then F is finite.*

Proof. Let $\mathcal{P}(\{1, 2, \ldots, n\})$ be the set of all subsets of $\{1, 2, \ldots, n\}$. Let $\phi : \mathcal{P}(\{1, 2, \ldots, n\}) \to \{1, 2, \ldots, 2^n\}$ be a bijective mapping.

Suppose $i \in \{1, 2, \ldots, 2^n\}$, $\pi_{A_1,\ldots,A_m} \in \Pi_n^m$. Let $\psi_i(\pi_{A_1,\ldots,A_m})$ be the number of sets A_j such that $\phi(A_j) = i$. Let $\omega : \Pi_n \to \mathbb{N}_0^{2^n}$,

$$\omega(\rho) = (\psi_1(\rho), \psi_2(\rho), \ldots, \psi_{2^n}(\rho)).$$

It can be easily checked that if $\omega(\rho_1) \leq \omega(\rho_2)$, then $\rho_1 \lesssim^2 \rho_2$. Therefore, the set $\{\omega(\rho) \mid \rho \in F\}$ is an antichain. By Lemma 4.32 this set is finite. Hence, the set F is also finite.

□

Lemma 4.34. *Suppose $F \subseteq \Pi_W$, F is an antichain. Then F is finite.*

Proof. Assume the converse. Let $\rho \in F \cap \Pi_n^m$. Since F is infinite, by Lemma 4.33 there exists $\sigma \in F \setminus \Pi_{m+n-2}$. It is easy to show that

$$\rho \lesssim \pi_{\{1,2,\ldots,n\},\underbrace{\varnothing,\varnothing,\ldots,\varnothing}_{m-1}} \lesssim \pi_{\{1,2,\ldots,m+n-1\}} \lesssim \sigma.$$

Hence $\rho \leq \sigma$. This contradiction completes the proof.

□

Lemma 4.35. *Suppose $\rho \in \Pi_n^m$, $\zeta(\rho) = (1, r)$, $r < n - 1$, then there exist $\rho' \in \Pi_{n-1}^{m+1}$ and $t > r$ such that $\rho' \lesssim \rho$ and $\zeta(\rho') = (1, t)$.*

Proof. Suppose $\rho = \pi_{A_1,\dots,A_m}$. Let S be the set of all $i \in \{1,\dots,n\}$ such that for every j we have $\{i\} \cup A_j \neq \{1,\dots,n\}$. Since $r < n-1$, we obtain that $|S| \geq 2$. Let $i_1, i_2 \in S$, $i_1 \neq i_2$. Without loss of generality it can be assumed that $i_1 = n-1$, $i_2 = n$. Put $A_i' = A_i \cap \{1,\dots,n-1\}$. Then

$$\pi_{A_1',\dots,A_m',\{1,2,\dots,n-2\}} \lesssim^3 \pi_{A_1',\dots,A_m',\varnothing} \lesssim^1 \pi_{A_1,\dots,A_m}.$$

It is easy to check that $\zeta(\pi_{A_1',\dots,A_m',\{1,2,\dots,n-2\}}) = (1,t)$, where $t > r$. This completes the proof.

□

Lemma 4.36. *Suppose $F \in \widetilde{\Pi}$, $F \not\subseteq (\Pi_n \cup \Pi_W)$ for every $n \in \mathbb{N}$. Then there exists an infinite antichain $G \subseteq F$.*

Proof. Let us construct an infinite sequence $\rho_1, \rho_2, \rho_3, \dots$ such that $\rho_i \in F$, $\mathrm{ar}(\rho_i) < \mathrm{ar}(\rho_{i+1})$ and $\zeta(\rho_i) > \zeta(\rho_{i+1})$ for every $i \in \{1,2,3,\dots\}$.

Let ρ_1 be an arbitrary predicate from $F \setminus \Pi_W$ such that $\zeta(\rho_1) = (1,t)$ for some $t \in \mathbb{N}$. Since $F \not\subseteq \Pi_W$ and F is a downset, ρ_1 exists. Suppose we already have $\rho_1, \rho_2, \dots, \rho_q$, and for every $i \in \{1,2,\dots,q\}$ we have $\zeta(\rho_i) = (1,t_i)$ for some $t_i \in \mathbb{N}$. Let $r = \mathrm{ar}(\rho_q)$. Let us define ρ_{q+1}.

Let $\sigma \in F \setminus (\Pi_W \cup \Pi_{3r})$. Suppose $\sigma \in \Pi_n^m$. Obviously, there exists $\sigma_0 \in \Pi_n^m$ such that $\sigma_0 \lesssim^3 \sigma$ and $\zeta(\sigma_0) = (1,s)$ for some s.

We have $n > 3r$. Using Lemma 4.35 several times for predicate σ_0 we get predicate σ' such that $\sigma' \lesssim \sigma_0$ and $\zeta(\sigma') = (1,t)$, where $t > r > t_q$. It is necessary to mention that if we obtain a predicate $\sigma' \in \Pi_{n'}^{m'}$ such that $\zeta(\sigma') = (1,t)$ and $t \geq n'-1$, then we do not apply Lemma 4.35 anymore. It is easy to check that $n' > 2r$ in this case.

Obviously, $\zeta(\sigma') < \zeta(\rho_q)$ and $\mathrm{ar}(\sigma') > \mathrm{ar}(\rho_q)$. Then we put $\rho_{q+1} = \sigma'$. It follows from Lemma 4.27 and Lemma 4.28 that predicates in this sequence are pairwise incomparable. This completes the proof.

□

Theorem 4.37. *Suppose $F \in \widetilde{\Pi}$, then*

$$|\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))| \begin{cases} < \infty, & \text{if } |F| < \infty; \\ = \aleph_0, & \text{if } |F| = \infty, F \subseteq (\Pi_n \cup \Pi_W) \text{ for some } n \in \mathbb{N}; \\ = 2^{\aleph_0}, & \text{otherwise.} \end{cases}$$

Proof. Suppose F is finite. By Theorem 4.31 we get $|\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))| < \infty$.

Suppose $F \not\subseteq (\Pi_n \cup \Pi_W)$ for every n. Then combining Lemma 4.36 and Theorem 4.31 we obtain that $|\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))| = 2^{\aleph_0}$.

Suppose there exists $n \in \mathbb{N}$ such that $F \subseteq \Pi_n \cup \Pi_W$. Then combining Lemma 4.33, Lemma 4.34 and Theorem 4.31 we get $|\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(F))| = \aleph_0$.

□

Theorem 4.38. *Suppose $M \in \Theta \cup \Phi$, then*

$$|\mathbb{L}_3^{\uparrow}(M)| \begin{cases} = \aleph_0, & \text{if } M \in \{\mathbf{C_2}, \mathbf{TC_2}, \mathbf{a}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}, \mathbf{aP}, \mathbf{aPN}, \\ & \mathbf{A}_\infty, \mathbf{A}_\infty\mathbf{M}, \mathbf{A}_\infty\mathbf{N}, \mathbf{AP}, \mathbf{APN}\} \\ & \text{or } M \in \bigcup\limits_{n\geq 1} \{\mathbf{aP_n}, \mathbf{AP_n}\}; \\ = 2^{\aleph_0}, & \text{if } M \in \{\mathbf{J_3}, \mathbf{aP}_\infty, \mathbf{aQ}, \mathbf{aW}, \mathbf{AP}_\infty, \mathbf{AQ}, \mathbf{AW}, \}; \\ < \infty, & \text{otherwise.} \end{cases}$$

Proof. For all clones except $\mathbf{aP_n}, \mathbf{AP_n}$ the proof follows from the description of the lattice and Theorem 4.37. The clones $\mathbf{aP_n}$ and $\mathbf{AP_n}$ are dual with respect to the transposition. Hence we consider only $\mathbf{aP_n}$.

By Theorem 4.12, $\mathbf{aP_n} \subset \mathrm{Clone}(F)$ iff $F \subseteq \Pi_n$. Hence, all clones from $\mathbb{L}_3^{\uparrow}(\mathbf{aP_n})$ except countable number belong to $\mathbb{L}_3^{\uparrow}(\mathrm{Clone}(\Pi_n))$. Therefore, by Theorem 4.37 we obtain that $\mathbb{L}_3^{\uparrow}(\mathbf{aP_n})$ is countable.

□

MAIN NOTATIONS

- $\mathbb{N} = \{1, 2, 3, \ldots\}$.
- $\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$.
- $E_k = \{0, 1, 2, \ldots, k-1\}$.
- $P_k^n = \{f \mid f : E_k^n \to E_k\}$.
- $P_k = \bigcup\limits_{n\geq 1} P_k^n$.
- $R_k^h = \{\rho \mid \rho : E_k^h \to \{0, 1\}\}$.
- $R_k = \bigcup\limits_{h\geq 0} R_k^h$.
- $|M|$ — the cardinality of the set M.
- $[M]$ — the closure of M.
- $\mathrm{Pol}(\rho)$ — the set of all functions $f \in P_k$ that preserve a predicate ρ.

- $\mathrm{Pol}(S) = \bigcap_{\rho \in S} \mathrm{Pol}(\rho)$.
- $\mathrm{Inv}(f)$ — the set of all predicates $\rho \in R_k$ that are preserved by a function f.
- $\mathrm{Inv}(M) = \bigcap_{f \in M} \mathrm{Inv}(f)$.
- ρ^* — the predicate that is dual to ρ with respect to the transposition.
- $\widetilde{\Pi}$ — the set of all nonempty downsets of Π.
- $\mathrm{Clone}(F) = \mathrm{Pol}\left(F \cup \{\rho_{+1}, \rho_W\}\right)$.
- $\mathrm{Clone}^*(F) = \mathrm{Pol}\left(F^* \cup \{\rho_{+1}, \rho_W^*\}\right)$.
- $\mathrm{Two}(a_1, \ldots, a_n)$ — the set of all elements that occur in the tuple $(a_1, \ldots, a_n)$ more than once.
- E_Π — the set of all equivalence classes generated by the quasiorder $\lesssim$ on the set Π.
- B_Π — the set of all antichains of E_Π excluding the one that consists of the bottom element $\widehat{\pi}_{\{\varnothing,\varnothing,\varnothing\}}$ only.
- $\mathrm{Bound}(F) := \{\widehat{\rho} \in E_\Pi \mid \widehat{\rho} \nsubseteq F, \forall \widehat{\sigma} \in E_\Pi (\widehat{\sigma} < \widehat{\rho} \Longrightarrow \widehat{\sigma} \subseteq F)\}$.
- $d(A) = min\{h \mid \exists Q \subseteq R_3^h : \mathrm{Pol}(Q) = A\}$.
- $\mathbb{L}_3 = \Theta \cup \Phi \cup \Upsilon$
- $\mathbb{L}_3^{\uparrow}(F) := \{F' \in \mathbb{L}_3 \mid F \subseteq F'\}$.
- $\mathbb{L}_3^{\downarrow}(F) := \{F' \in \mathbb{L}_3 \mid F' \subseteq F\}$.
- $\mathrm{ar}(\rho)$ — the arity of a predicate ρ.
- $\mathrm{VarValues}(\rho, i) = \{\alpha(i) \mid \alpha \in \rho\}$.
- $\mathrm{Shift}(\rho)$ — the set of all predicates that can be obtained from ρ by shifting and permutation of variables.
- $\mathrm{Shift}(S) = \bigcup_{\rho \in S} \mathrm{Shift}(\rho)$.
- $\mathrm{And}(S)$ — the set of all $\rho \in R_k$ that can be presented as a conjunction of predicates from S.

- $\mathrm{Strike}(\rho, i)$ — the predicate that is obtained from ρ by striking the i-th row.

- $\mathrm{Strike}(\rho)$ — the set of all predicates that can be obtained from ρ by striking rows.

- $\mathrm{Strike}(S) = \bigcup\limits_{\rho \in S} \mathrm{Strike}(\rho)$.

- $|\alpha|$ — the length of α.

- $[_l(\alpha) = \alpha(|\alpha| - l + 1) \ldots \alpha(|\alpha| - 1)\alpha(|\alpha|)$.

- $]_l(\alpha) = \alpha(1)\alpha(2) \ldots \alpha(l)$.

- $\alpha^s = \underbrace{\alpha\alpha \ldots \alpha}_{s}$.

- $B_0 = \{false, true, \rho_{+1}, \sigma_3^{=}, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}\}$.

- $B_1 = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \rho_{\leq}, \rho_{\vee,2}, \rho_N, \rho_W \right\}$.

- B_2 — the set of all predicates $\rho \in \mathrm{Inv}(right)$ such that for every $i \in \{1, 2, \ldots, \mathrm{ar}(\rho)\}$ we have $\mathrm{VarValues}(\rho, i) \subseteq \{0, 1\}$.

- $Main$ — set of all predicates $\rho \in R_3$ such that the following conditions hold for some $m \in \{1, \ldots, \mathrm{ar}(\rho)\}$:

  1. $\mathrm{VarValues}(\rho, i) \subseteq \{0, 1\}$ for every $i \in \{1, 2, \ldots, m\}$.
  2. For every $a_{m+1}, \ldots, a_{\mathrm{ar}(\rho)} \in E_3$ we have

  $$\rho(1, \ldots, 1, a_{m+1}, \ldots, a_{\mathrm{ar}(\rho)}) = 1.$$

- $\sigma_k^{=} = \begin{pmatrix} 0 & 1 & \ldots & k \\ 0 & 1 & \ldots & k \end{pmatrix}$

- $false$ — the predicate of arity 0 that takes on value 0.

- $true$ — the predicate of arity 0 that takes on value 1.

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